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OPTIMAL STATION KEEPING AT
THE L4 LIBRATION POINT

THESIS

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THE L4 LIBRATION POINT

THESIS

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Preface

This paper describes a solution to the problem of optimally controlling the drift of a space vehicle stationed at one of the earth-moon triangular libration points. Many libration point studies have preceded this one and I am greatly indebted to the foundation which they have provided. Of particular importance to this study were the investigations by Dr. Lynn E. Wolaver and other studies performed under Dr. Wolaver's sponsorship, as described in Chapter I of this report.

This thesis began as a follow-on study to the analog computer investigation by Captain Gerald T. Rudolph (Ref 2). However, after much effort, it became evident that using a trial and error approach in the style of Captain Rudolph, I could not hope to obtain satisfactory results. This approach was finally abandoned in favor of modern optimal control theory techniques. The change was made with reservation due to the belief that a satisfactory formulation and solution in this manner might not be possible. Fortunately, I was able to formulate the problem in a manner which made it readily solvable on a high-speed digital computer.

I wish to express my appreciation to my thesis advisor, Captain Edward A. Kern, for his support and advice. I also want to thank Dr. Lynn E. Wolaver of the Aerospace Research Laboratories, who proposed and sponsored this thesis, for his help and suggestions.

Contents

Page

Preface ii

List of Figures v

List of Tables vi

List of Symbols vii

Abstract ix

I. Introduction 1

Background 1
The Very Restricted Four-Body Problem . . 1
Minimizing Drift by Selection of Initial Conditions 6
Analog Simulation 6
Known Cost 6
Problem Formulation 8
Performance Index 8
System Equations 8
Approach 10

II. Analysis 12

Theory 12
Application 15

III. Digital Computations 21

The Gain Matrix 21
System Performance 24

IV. Analog Simulation 32

Steady-State Tendency 32
Initial Conditions for Minimum Cost 34
Optimum Initial Conditions without Control . 37

V. Discussion 55

Model Deficiencies 55
Computational Approximations 55
Realizability 55
Thrust Implimentation 55
Sensor Requirements 56
Attitude Control 56

Contents

Page

VI. Summary and Recommendations 58

Summary 58
Recommendations 58

Bibliography 60

Appendix A: Fortran Program for Gain Calculations . . 61

Appendix B: Fortran Program for Performance Calculations 65

Appendix C: Analog Diagrams 78

List of Tables

Table		Page
I	Steady-State Gains	25
II	Steady-State Gains (continuation)	26
III	Performance of the Optimal Control	27
IV	Performance of the Modified Control	28
V	Comparison - Optimal vs Modified Control	29
VI	Digital Data - "Steady-State", q=0.01	51
VII	Digital Data - "Steady-State", q=0.02	52
VIII	Digital Data - "Steady-State", q=0.03	53

List of Tables

Table		Page
I	Steady-State Gains	25
II	Steady-State Gains (continuation)	26
III	Performance of the Optimal Control	27
IV	Performance of the Modified Control	28
V	Comparison - Optimal vs Modified Control	29
VI	Digital Data - "Steady-State", q=0.01	51
VII	Digital Data - "Steady-State", q=0.02	52
VIII	Digital Data - "Steady-State", q=0.03	53

List of Symbols

$a, b, c, \cdots h$	constants of the system equations
m	mass of the moon in normalized units
n	the constant M_s/R^3 in normalized units
q	drift weighting factor of the quadratic performance criterion
t	time
t_0	initial time
W	radian frequency difference $\omega - \Omega$
x, y	rotating coordinate system with origin at L4
$C_1, C_2, \cdots C_{12}$	constants of linearized equations of motion
$L_1, L_2, \cdots L_5$	the restricted three-body libration points
M_e, M_m, M_s	masses of the earth, moon, and sun respectively
P	force function
R	distance from the sun to the earth-moon barycenter
T	final time
X_L, Y_L	location of L4 relative to the earth-moon barycenter
A, B	matrices of the system equations
F, Q, R	matrices of the quadratic performance criterion
J	quadratic performance criterion
K	gain matrix
$\underline{X}$	the state vector
$\underline{U}$	the control vector (thrust acceleration)
A_{ij}	element of the A matrix (i^{th} row, j^{th} column)
K_{ij}	element of the K matrix (i^{th} row, j^{th} column)

List of Symbols

$X_1, X_2, \cdots X_7$	state variables
U_x, U_y	components of the control vector
U_{av}	time-average magnitude of the control vector
ΔV	total velocity increment
I_{sp}	specific impulse
g_o	sea level gravity
ϕ	direction of the sun relative to the x-y coordinate system
α	initial direction of the sun relative to the x-y coordinate system
ω	angular velocity of the moon in orbit about the earth-moon barycenter
Ω	angular velocity of the earth-moon barycenter in orbit about the sun
$\cdot$	placed above a variable to denote differentiation with respect to time
$'$	a superscript used to denote the transpose of a matrix
-1	a superscript used to denote the inverse of a matrix
$\langle , \rangle$	denotes an inner product

Abstract

A control system is devised for maintaining a space vehicle in close proximity to one of the earth-moon triangular libration points while minimizing fuel consumption. The problem is formulated as an optimal state regulator problem of variational calculus and then modern control theory is applied. A quadratic performance criterion is used which leads to a linear feedback control system. The feedback gains are obtained by solving a matrix differential equation of the Ricatti type on a high-speed digital computer. Performance of the resulting optimal system is verified and further analyzed on an analog computer.

OPTIMAL STATION KEEPING AT THE L4 LIBRATION POINT

I. Introduction

Because of the strong perturbing influence of the sun, a space vehicle stationed at one of the triangular Lagrange libration points of the earth-moon system would require a thrust control system to confine its drift. It is the object of this thesis to devise a realizable control system which uses a minimum of fuel to restrict the drift of a space vehicle stationed near one of these libration points.

Background

The Very Restricted Four-Body Problem. The libration points are equilibrium points at which a particle would, if given the proper initial velocity, remain fixed relative to the two other masses of a restricted three-body system. The adjective "restricted" imposes two conditions; the particle's mass must be so small that it does not affect the motion of the two larger masses and the two larger masses must move in circular orbits about their common center of mass. Lagrange, in 1772 proved that five such points exist. Their locations, fixed in a rotating coordinate system, are illustrated in Figure 1. It is well established for this three-body system that if the mass ratio of the two large bodies is less than 0.04, then L4 and L5 are stable in the sense that a small disturbance results in a non-divergent motion

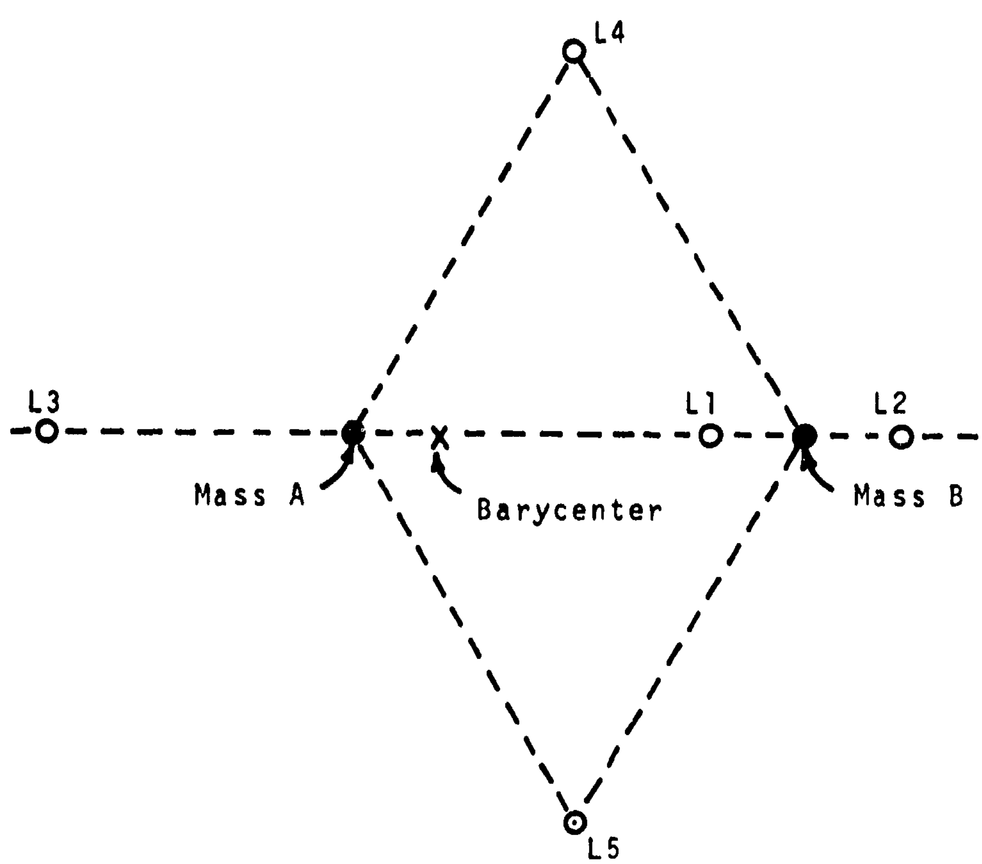


Figure 1

LIBRATION POINTS OF THE RESTRICTED THREE-BODY PROBLEM

close to the libration point. The other three equilibrium points are unstable for all mass ratios.

For the case of the earth-moon system, the effect of a fourth body, the sun, can not be ignored. A mathematical model which has been used successfully to represent the earth-moon-sun system is the model known as "the very restricted four-body problem" (Refs 2, 3, 4, & 5). This model retains the conditions of the restricted three-body problem and imposes the additional conditions that the center of mass of the earth-moon system move in a circular orbit about the center of mass of the entire system and that the orbits be coplaner (see Figure 2). Equations of motion for this model have been derived and nondimensionalized (Refs 3 and 4) by setting the unit of distance equal to the earth-to-moon distance, the unit of mass equal to the total mass of the earth and moon, and the unit of time such that ω is unity and one time unit is equal to 4.348 mean solar days. The nondimensional equations are

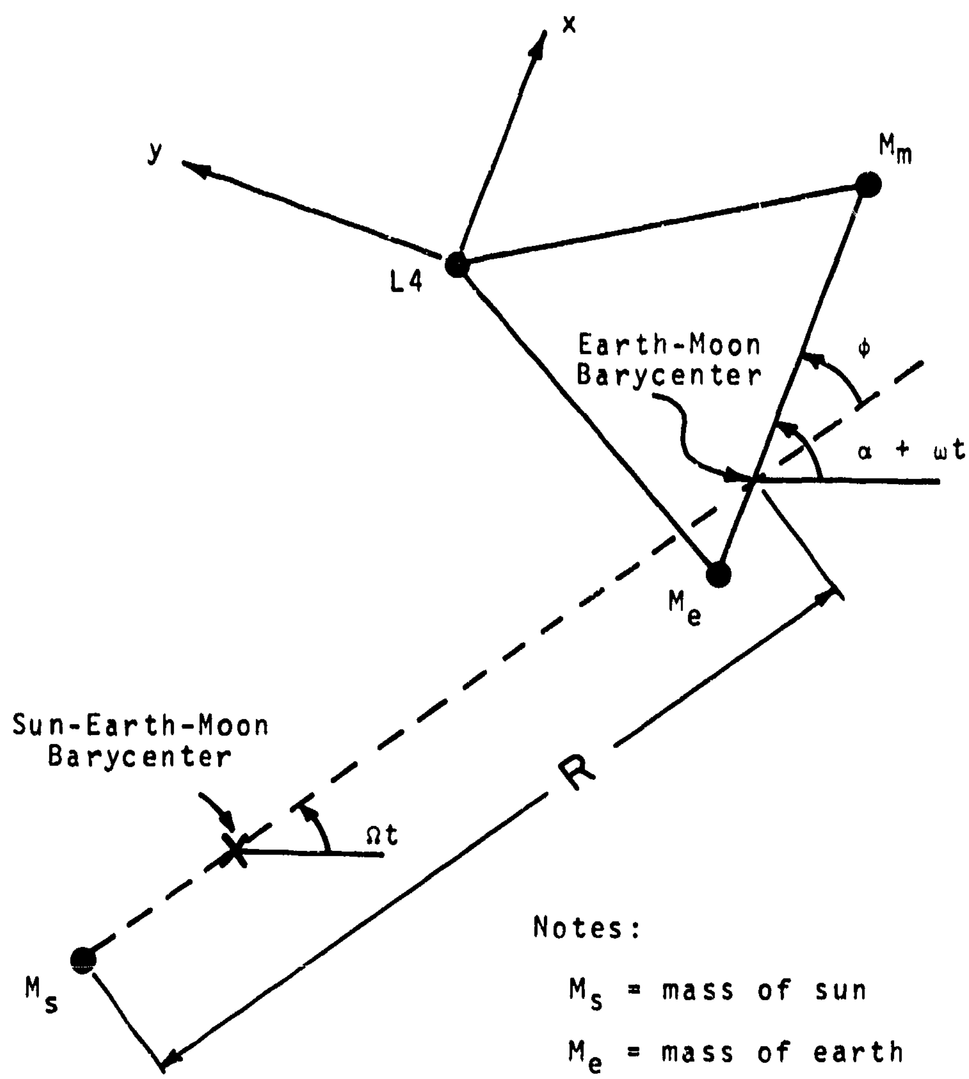
$$\ddot{x} - 2\dot{y} - x - (M_S/R^2) \cos \phi = \partial P/\partial x \qquad (1)$$

$$\ddot{y} + 2\dot{x} - y + (M_S/R^2) \sin \phi = \partial P/\partial y \qquad (2)$$

in which

$$P = \frac{m}{[(x - 1 + m)^2 + y^2]^{\frac{1}{2}}} + \frac{1 - m}{[(x + m)^2 + y^2]^{\frac{1}{2}}}$$

$$+ \frac{M_S}{[(x + R \cos \phi)^2 + (y - R \sin \phi)^2]^{\frac{1}{2}}}$$



Notes:

M_s = mass of sun

M_e = mass of earth

M_m = mass of moon

$\phi = \alpha + (\omega - \Omega)t$

Figure 2

VERY RESTRICTED FOUR-BODY REPRESENTATION

OF THE EARTH-MOON-SUN SYSTEM

$\phi = wt + \alpha$

$m = M_m$ (in nondimensional mass units)

$w = (\omega - \Omega)/\omega$

When linearized about the old L4 libration point the equation becomes (Refs 5:80-81 and 4:37-39)

$$\begin{aligned}\ddot{x} - 2\dot{y} - C_1 x - C_2 y = {} & C_4 + C_6 \cos 2\phi + C_7 \sin 2\phi \\ & + C_8 \cos \phi + C_9 \sin \phi \\ & + C_{11} n\,(x \cos 2\phi - y \sin 2\phi) \\ & + C_{12} n\,(3X_L \cos \phi - Y_L \sin \phi)x \\ & + C_{12} n\,(Y_L \cos \phi - X_L \sin \phi)y \end{aligned} \tag{3}$$

$$\begin{aligned}\ddot{y} + 2\dot{x} - C_2 x - C_3 y = {} & C_5 + C_7 \cos 2\phi - C_6 \sin 2\phi \\ & - C_9 \cos \phi + C_{10} \sin \phi \\ & - C_{11} n\,(x \sin 2\phi + y \cos 2\phi) \\ & + C_{12} n\,(Y_L \cos \phi - X_L \sin \phi)x \\ & + C_{12} n\,(X_L \cos \phi - 3Y_L \sin \phi)y \end{aligned} \tag{4}$$

in which (in normalized units)

$$\begin{aligned} X_L &= (1 - 2m)/2 & C_7 &= -3nY_L/2 \\ Y_L &= \sqrt{3}/2 & C_8 &= -3n(3X_L^2 + Y_L^2)/8R \\ n &= M_s/R^3 & C_9 &= 3nX_L Y_L/4R \\ C_1 &= (3 + 2n)/4 & C_{10} &= 3n(3Y_L^2 + X_L^2)/8R \\ C_2 &= 3\sqrt{3}\,(1 - 2m)/4 & C_{11} &= 3/2 \\ C_3 &= (9 + 2n)/4 & C_{12} &= -3/4R \\ C_4 &= nX_L/2 \\ C_5 &= nY_L/2 \\ C_6 &= 3nX_L/2 \end{aligned} \tag{5}$$

Minimizing Drift by Selection of Initial Conditions. In a Masters Thesis by Capt. Paul M. Ulshafer (Ref 4) and in a study by Dr. Lynn E. Wolaver (Ref 5), initial conditions were derived which minimized the drift of a space vehicle stationed near the L4 point. The studies were based on the model given by the very restricted four-body problem. It was found that by making a judicious choice of initial conditions, which include the initial sun direction α, the drift from the L4 point remains less than 0.013 distance units during a one-year mission.

Analog Simulation. In a Masters Thesis by Capt. Gerald T. Rudolph (Ref 2), an attempt was made using an analog computer to "minimize the energy required to hold a space vehicle at the earth-moon L4 libration point." The study failed to accomplish this stated intention, however, it did demonstrate the following:

(1) the feasibility of using an analog computer to simulate the equations of motion.

(2) the accuracy of using the linearized equations, Eqs (3) and (4), without the terms involving C_8, C_9, C_{10}, and C_{12}.

Known Cost. Prior to this investigation, only the two extremes of L4 station keeping cost were known, that for no thrust control and that for perfect control. Without thrust control, thrust cost is of course zero; however, the cost in drift would be, assuming optimum initial conditions, 0.013 for a one-year mission. Perfect control, i.e., continuous

maintenance of the satellite precisely at L4, would have no cost in drift, however, the cost in thrust would be finite for any given mission duration. The thrust acceleration required for perfect control would be the amount necessary to exactly cancel the forcing function of Eqs (3) and (4), i.e.,

$$U_x = -(C_4 + C_6 \cos 2\phi + C_7 \sin 2\phi + C_8 \cos \phi + C_9 \sin \phi) \tag{6}$$

$$U_y = -(C_5 + C_7 \cos 2\phi - C_6 \sin 2\phi - C_9 \cos \phi + C_{10} \sin \phi) \tag{7}$$

The average magnitude of the required thrust acceleration would be

$$U_{av} = (1/2\pi) \int_0^{2\pi} (U_x^2 + U_y^2)^{\frac{1}{2}} \, d\phi \tag{8}$$

To compute U_{av} analytically would be difficult, therefore numeral integration using the digital computer was employed. The terms containing C_8, C_9, and C_{10} were neglected giving an approximate result (in normalized units) of

$$U_{av} = 0.008576 \tag{9}$$

Between the two extremes of permitting no drift and permitting an uncontrolled drift, the control cost was unknown prior to this study. For all that was known at the time, the optimum thrust policy might have been to maintain perfect control.

Problem Formulation

Performance Index. The purpose of this thesis is to minimize the fuel requirements of a space vehicle stationed near the earth-moon L4 libration point. Although only the L4 point is considered, the results apply equally well to the L5 point because of symmetry.

In order that the study results be independent of vehicle weight or rocket performance, total velocity increment, ΔV, is used as a cost measure rather than fuel. ΔV is calculated from the relation

$$\Delta V = \int_0^t \|\underline{U}(t)\| \, dt \tag{10}$$

in which $\underline{U}(t)$ is the acceleration due to the thrust. ΔV is related to fuel according to the well-known rocket equation

$$\text{Fuel Weight} = [\text{Final Vehicle Weight}] \, [\exp(\Delta V/g_0 I_{sp}) - 1] \tag{11}$$

Therefore, if it is assumed that I_{sp} is constant and that no staging occurs, minimum ΔV implies minimum fuel. The performance index is then the ΔV required to hold the satellite within some given peak drift distance for a given amount of time. No consideration will be given to attitude control costs.

System Equations. The model of the Very Restricted Four-Body Problem is assumed in this study. If the terms involving C_8, C_9, C_{10}, and C_{12} are dropped from the linearized equations and a thrust acceleration term is added,

nondimensionalized system equations of sufficient accuracy are obtained:

$$\ddot{x} - 2\dot{y} - C_1 x - C_2 y = C_4 + C_6 \cos 2\phi + C_7 \sin 2\phi$$

$$+ C_{11} n \,(x \cos 2\phi - y \sin 2\phi) + U_x \qquad (12)$$

$$\ddot{y} + 2\dot{x} - C_2 x - C_3 y = C_5 + C_7 \cos 2\phi - C_6 \sin 2\phi$$

$$- C_{11} n \,(x \sin 2\phi + y \cos 2\phi) + U_y \qquad (13)$$

The coefficients were evaluated based on the astronomical constants (in normalized units)

$$1/m = 82.301$$
$$M_S = 328{,}905.2$$
$$\Omega = 0.07480133 \qquad (14)$$

resulting in the values

$$w = 0.9252$$
$$X_L = 0.4878$$
$$Y_L = 0.8660$$
$$n = 0.005595$$
$$C_1 = 0.7528$$
$$C_2 = 1.267$$
$$C_3 = 2.253$$
$$C_4 = 0.001365$$
$$C_5 = 0.002423$$
$$C_6 = 0.004094$$
$$C_7 = -\,0.007268$$
$$C_{11} = 3/2 \qquad (15)$$

Unless otherwise noted, all quantities are expressed in normalized units. The normalized units can be converted into physical units using the following relations:

$$
\begin{aligned}
&1 \text{ time unit} = 4.348 \text{ mean solar days} \\
&1 \text{ distance unit} = 239{,}000 \text{ statute miles} \\
&1 \text{ velocity unit} = 3360 \text{ ft/sec} \\
&1 \text{ acceleration unit} = 0.008943 \text{ ft/sec}^2
\end{aligned}
\tag{16}
$$

Approach

It was believed at first that a trial and error approach using an analog computer would yield a solution to the problem. Several thrusting schemes were postulated and tested which did actually limit the drift to any desired value for any duration. However, measurement of the cost (ΔV) indicated no saving when compared to the cost of perfect control. One of the schemes tested used a control design based on a root locus analysis of the transfer function x/U_y. Beautiful stability resulted, however, cost measurements could not prove its merit.

The problem was successfully solved by considering it as an optimal regulator problem of variational calculus. The direct application of modern control theory is at best a difficult task due to several peculiarities of the problem. These are:

(1) the constraint of peak drift,

(2) the time varing coefficients of the system equations and

(3) the absence of a terminal cost or end-point conditions.

The approach used, therefore, was to first devise a substitute performance criterion which could be handled by Pontryagin's maximum principle, solve for the optimal control, and finally measure its worth in terms of the actual performance criterion. It was decided to use, without constraints, the quadratic cost function

$$J = \frac{1}{2} \int_0^T [q\,(x^2 + y^2) + U_x^2 + U_y^2]dt \qquad (17)$$

hopeful that perhaps the drift could be controlled to any desired peak value merely by varying the weighing factor q. This particular performance index was chosen for two very important reasons:

(1) it leads to a readily solvable control law

(2) the control turns out to be a linear function of state, and hence a closed-loop control system results.

A closed-loop control system is highly desirable since the system equations do not take into account the real-world perturbations or even the nonlinearities of the very restricted four-body problem. With a feedback (closed loop) control it becomes more likely that such perturbations have little effect.

The remainder of this report describes the technique and results of taking this final approach to the problem.

II. Analysis

Theory

For a linear system, the optimal control with respect to a quadratic criterion can be obtained from a rather eloquent theory. It is eloquent not only from a mathematical point of view but also from an engineering point of view, for the optimal control turns out to be a linear function of state. The theory was developed by Kalman, but a derivation based on Pontryagin's maximum principle can be found in Optimal Control by Athans and Falb (Ref 1:750-766). A summary of the control law which results from this theory follows (Ref 1:762-763):

Given the linear system

$$\dot{\underline{X}}(t) = A(t)\underline{X}(t) + B(t)\underline{U}(t) \tag{18}$$

and the cost functional

$$J = \frac{1}{2}\langle \underline{X}(T),\ F\ \underline{X}(T)\rangle + \frac{1}{2}\int_{t_0}^{T} [\langle \underline{X}(t),\ Q(t)\underline{X}(t)\rangle + \langle \underline{U}(t),\ R(t)\underline{U}(t)\rangle]dt \tag{19}$$

where $\underline{U}(t)$ is not constrained, T is specified, F and $Q(t)$ are positive semidefinite, and $R(t)$ is positive definite. Then an optimal control exists, is unique, and is given by the equation

$$\underline{U}(t) = -R^{-1}(t)B'(t)K(t)\underline{X}(t) \tag{20}$$

where the n x n symmetric matrix $K(t)$ is the unique solution of the Riccati equation

$$\dot{K}(t) = -K(t)A(t) - A'(t)K(t) + K(t)B(t)R^{-1}(t)B'(t)K(t) - Q(t) \qquad (21)$$

satisfying the boundary condition

$$K(T) = F \qquad (22)$$

The state of the optimal system is then the solution of the linear differential equation

$$\dot{\underline{X}}(t) = [A(t) - B(t)R^{-1}(t)B'(t)K(t)]\,\underline{X}(t) \qquad (23)$$

Figure 3 shows the structure of the optimal system. Note that the optimal control is generated by a linear, time-varying, feedback gain. Since $R(t)$ and $B(t)$ are known matrices, it follows that the matrix $K(t)$ determines the gain of the feedback path and the behavior of the system. The matrix $K(t)$ is therefore referred to as the "gain" matrix.

The gain matrix is the solution of a nonlinear matrix equation, Eq (21), and for this reason a closed form solution is not usually obtainable. This is no great obstacle however, since a numerical solution of the gain matrix can be obtained easily using a digital computer. Note that Eq (21) must be solved backward in time from time T in order that the boundary condition given by Eq (22) is satisfied.

One fortunate result of the theory is that the gain matrix $K(t)$ is independent of the state. Once the system, the cost J, and the terminal time T have been specified,

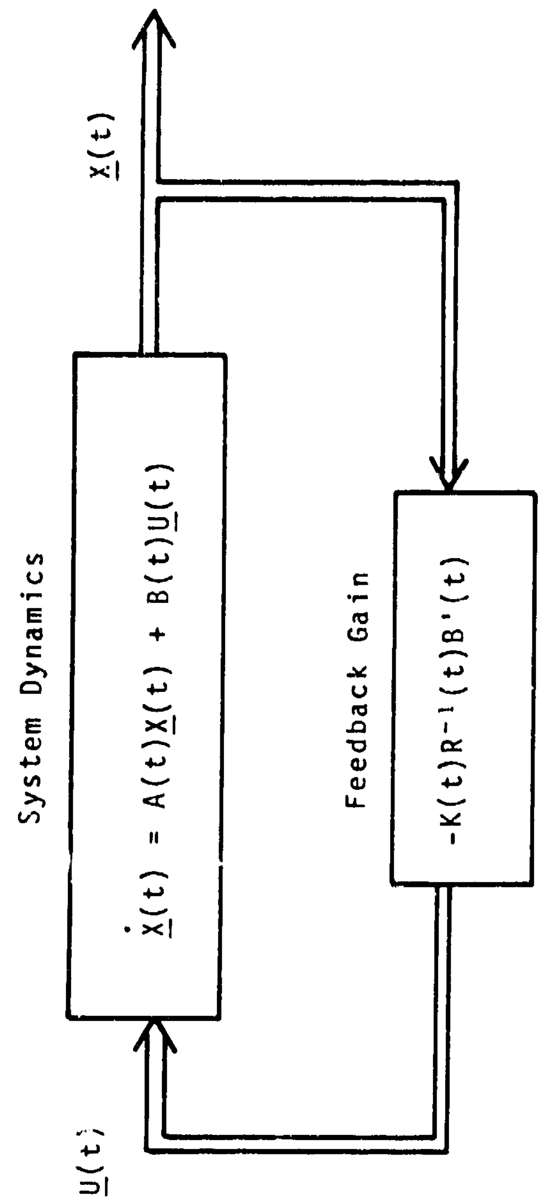


Figure 3

STRUCTURE OF THE OPTIMAL SYSTEM

$K(t)$ can be computed. Initial conditions of the states have no effect on the design of the optimal control system. The initial conditions affect only the cost.

Application

The system equations used in this analysis were presented in the introduction (see page 9). For ease of reference they are repeated here in a slightly different form:

$$\ddot{x} = 2\dot{y} + ax + by + h\,(x \cos 2\phi - y \sin 2\phi)$$
$$+ d + f \cos 2\phi - g \sin 2\phi + U_x \qquad (24)$$

$$\ddot{y} = -2\dot{x} + bx + cy - h\,(x \sin 2\phi + y \cos 2\phi)$$
$$+ e - g \cos 2\phi - f \sin 2\phi + U_y \qquad (25)$$

where

$\phi = wt + \alpha$

α = initial direction of the sun

$w = 0.9252$

$a = C_1 = 0.7528$

$b = C_2 = 1.267$

$c = C_3 = 2.253$

$d = C_4 = 0.001365$

$e = C_5 = 0.002423$

$f = C_6 = 0.004094$

$g = -C_7 = 0.007268$

$h = C_{11}n = 0.008393 \qquad (26)$

Before the control law can be applied, the system equations must be put in the form of Eq (18), that is of the form

$$\dot{\underline{X}}(t) = A(t)\underline{X}(t) + B(t)\underline{U}(t) \tag{18}$$

It is not clear, at first glance, that this task is possible due to the time varying forcing functions which appear in the system equations. However, by defining an extra three state variables (X_5, X_6, and X_7), the necessary formulation can be accomplished.

The state variables are defined as follows

$$\begin{aligned} X_1 &= x \\ X_2 &= \dot{x} \\ X_3 &= y \\ X_4 &= \dot{y} \\ X_5 &= 1 \\ X_6 &= \cos 2\phi \\ X_7 &= \sin 2\phi \end{aligned} \tag{27}$$

They are inter-related by the following equations:

$$\begin{aligned} \dot{X}_1 &= X_2 \\ \dot{X}_2 &= 2X_4 + (a + h \cos 2\phi)X_1 \\ &\quad + (b - h \sin 2\phi)X_3 + dX_5 \\ &\quad + fX_6 - gX_7 + U_x \\ \dot{X}_3 &= X_4 \end{aligned}$$

$$\dot{X}_4 = -2X_2 + (b - h \sin 2\phi)X_1$$
$$+ (c - h \cos 2\phi)X_3 + eX_5$$
$$- gX_6 - fX_7 + U_y$$
$$\dot{X}_5 = 0$$
$$\dot{X}_6 = -2wX_7$$
$$\dot{X}_7 = 2wX_6 \tag{28}$$

The system equations are now expressable in the required form, i.e.

$$\underline{\dot{X}}(t) = A(t)\underline{X}(t) + B(t)\underline{U}(t) \tag{18}$$

in which

$$\underline{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ X_4(t) \\ X_5(t) \\ X_6(t) \\ X_7(t) \end{bmatrix} \tag{29}$$

$$\underline{U}(t) = \begin{bmatrix} U_x(t) \\ U_y(t) \end{bmatrix} \tag{30}$$

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ (a + h \cos 2\phi) & 0 & (b - h \sin 2\phi) & 2 & d & f & -g \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ (b - h \sin 2\phi) & -2 & (c - h \cos 2\phi) & 0 & e & -g & -f \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2w \\ 0 & 0 & 0 & 0 & 0 & 2w & 0 \end{bmatrix} \tag{31}$$

$$B(t) = B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{32}$$

The cost functional used in this analysis, as discussed previously (see page 11) is the quadratic cost

$$J = \frac{1}{2} \int_0^T [q\,(X_1^2 + X_3^2) + (U_x^2 + U_y^2)]dt \tag{33}$$

which can be expressed in the form of Eq (19)

$$J = \frac{1}{2} \langle \underline{X}(T),\ F\ \underline{X}(T) \rangle + \frac{1}{2} \int_{t_0}^T [\langle \underline{X}(t),\ Q(t)\underline{X}(t) \rangle + \langle \underline{U}(t),\ R(t)\underline{U}(t) \rangle]dt \tag{19}$$

by letting

$$F = [0] \quad \text{a 7 by 7 zero matrix} \tag{34}$$

$$Q(t) = Q = \begin{bmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

$$R(t) = R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{36}$$

$$t_0 = 0 \tag{37}$$

Equations (20) and (23) can now be simplied somewhat by carrying out the indicated matrix multiplications. The following equations result:

$$\underline{U}(t) = - \begin{bmatrix} K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} \end{bmatrix} \underline{X}(t) \tag{38}$$

$$\dot{\underline{X}}(t) = \left\{ A(t) \dot{-} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\} \underline{X}(t) \tag{39}$$

where the gain matrix is defined as

$$K(t) = \begin{bmatrix} K_{11} & K_{12} & \cdot & \cdot & \cdot & \cdot & K_{17} \\ K_{21} & K_{22} & \cdot & \cdot & \cdot & \cdot & K_{27} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{71} & K_{72} & \cdot & \cdot & \cdot & \cdot & K_{77} \end{bmatrix} \tag{40}$$

This completes the adaptation of the problem to fit the control law stated in the previous section (see page 12). Note that the problem now satisfies all the necessary conditions required by the control law. Eq (21) must now be solved (by some numerical technique) for the elements of the gain matrix. Then Eqs (38) and (39) can be used to obtain the optimal control and the system performance.

III. Digital Computations

The nature of the gain matrix and the performance of the system is investigated in this section. The gain matrix and the system performance was computed with the aid of an IBM 7094 digital computer using the Runge-Kutta method of numerical integration.

In the course of this investigation it becomes evident that modifications can be made in the application of the control law. These modifications are motivated by the desire to reduce the complexity of the control system. Fortunately, several other considerations lend support to the same modifications, namely considerations of the ease of computation, the ease of making an analog simulation, and improved performance in terms of the actual cost of the system (see page 8).

The Gain Matrix

Figure 4 is a sketch which illustrates the nature of the gain matrix elements. In order to show all characteristics in a single illustration, a composite of the gain matrix elements is depicted rather than any one particular element. Each gain matrix element displays the same general trend, that of remaining at some steady state level with a slight oscillation until the terminal time T is approached, then decaying to zero. The elements differ in the oscillation amplitude in the steady state region, the point at which the decay begins, and the character (smooth or oscillatory) of

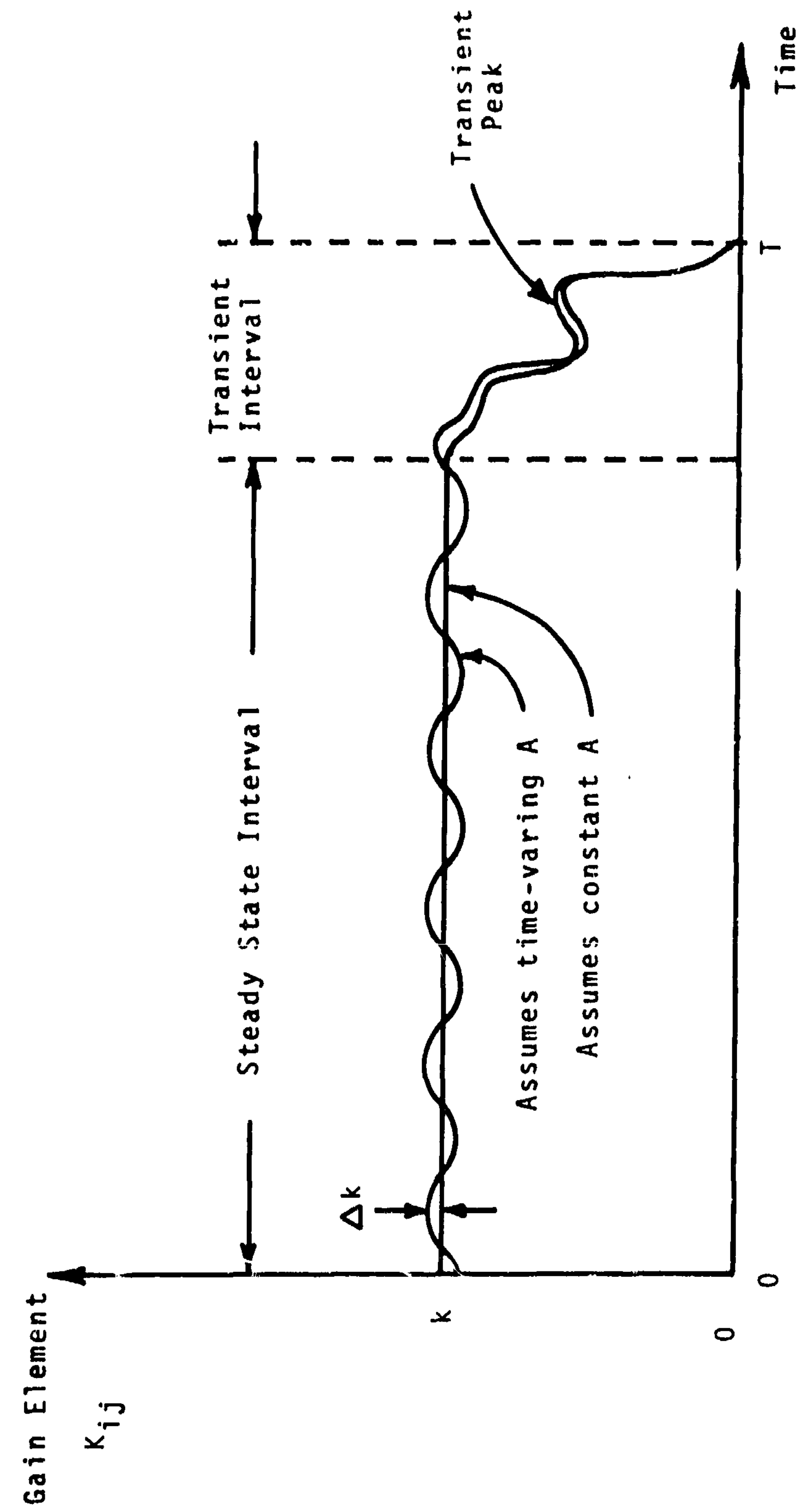


Figure 4

A GAIN MATRIX ELEMENT (COMPOSITE)

the decay.

Three traits of the gain elements become less pronounced as q, weighting on drift, is increased. These traits are: oscillations during the steady state interval, peaks during the transient interval, and a relatively long duration transient interval. As q is increased, each of the gain elements exhibit more nearly a step function character, as viewed backwards in time. That is, the oscillation amplitudes decrease, the peaks in the transient interval diminish, and the transient interval becomes very small. For a q of 0.1, the most severe oscillation is only about 6% (Δk relative to k), exhibited by element K_{26}. This element also exhibits the most severe transient interval, about twenty normalized time units for a q of 0.1. Since the oscillations are, in general, of such small magnitudes and because the transient intervals are generally short relative to a one-year mission (84 normalized time units), a modification of the control law is considered which ignores the transient interval and the oscillation. The modified control law differs from the optimal law only in that it uses a constant gain matrix rather than the time-varying one. The constant gain matrix is set equal to the average steady-state level of the time-varying gain matrix. System performance using this modified control law is described in the next section.

The constant gain matrix is somewhat less difficult to compute than the time-varying gain matrix. Also, the computation no longer depends upon the terminal time T. A

constant "A" matrix whose elements are the average values of the time-varying A matrix elements is used for the computation. The time-varying elements of the A matrix are

$$
\begin{aligned}
A_{21} &= 0.7528 + 0.008393 \cos 2\phi \\
A_{23} &= 1.267 - 0.008393 \sin 2\phi \\
A_{41} &= A_{23} \\
A_{43} &= 2.253 - 0.008393 \cos 2\phi
\end{aligned} \qquad (41)
$$

The approximation is therefore a very close one since the time-varying values never differ from their average values by more than 0.008393. The results of calculating the gain matrix in this fashion are tabulated in Tables I and II for values of q ranging from .01 to 1,000,000. Only those elements required by Eqs (38) and (39) are listed. The computer program used to compute this data is described in Appendix A.

System Performance

System performance using the modified control law was investigated and compared with performance of the optimal system. Results of this comparison show that the modified control system is superior to the optimal system when actual costs are considered rather than quadratic cost.

System performance data was computed for the optimal control and the modified control systems using a q of 1.0 and a mission time of 15 (about two months). This data is listed, for the last 0.4 time units of the mission, in Tables III and IV. Peak drift with the optimal control

Table I

STEADY-STATE GAINS

q	K_{21}	K_{22}	K_{23}	K_{24}	K_{25}	K_{26}	K_{27}
0.0100	-0.6512	0.5588	-1.1179	-0.4555	-0.0012171	-0.0007457	-0.0042003
0.0200	-0.7096	0.6090	-1.2283	-0.5134	-0.0013314	-0.0005891	-0.0047957
0.0300	-0.7288	0.6346	-1.2759	-0.5372	-0.0013760	-0.0005028	-0.0051252
0.0400	-0.7342	0.6519	-1.3013	-0.5485	-0.0013962	-0.0004636	-0.0053579
0.0500	-0.7329	0.6653	-1.3161	-0.5537	-0.0014045	-0.0004367	-0.0055421
0.0600	-0.7280	0.6764	-1.3250	-0.5555	-0.0014064	-0.0004199	-0.0056976
0.0700	-0.7209	0.6862	-1.3303	-0.5552	-0.0014044	-0.0004090	-0.0058342
0.0800	-0.7124	0.6951	-1.3332	-0.5536	-0.0013998	-0.0004015	-0.0059573
0.0900	-0.7030	0.7032	-1.3346	-0.5511	-0.0013935	-0.0003958	-0.0060702
0.1000	-0.6931	0.7109	-1.3348	-0.5479	-0.0013861	-0.0003912	-0.0061752
0.2000	-0.5855	0.7730	-1.3151	-0.5062	-0.0012914	-0.0003304	-0.0069776
0.3000	-0.4829	0.8233	-1.2894	-0.4649	-0.0011963	-0.0002149	-0.0075401
0.4000	-0.3890	0.8675	-1.2670	-0.4286	-0.0011097	-0.0000643	-0.0079669
0.5000	-0.3027	0.9078	-1.2485	-0.3967	-0.0010312	0.0001022	-0.0083036
0.6000	-0.2227	0.9449	-1.2332	-0.3685	-0.0009595	0.0002736	-0.0085663
0.7000	-0.1479	0.9796	-1.2207	-0.3433	-0.0008937	0.0004435	-0.0087803
0.8000	-0.0776	1.0123	-1.2104	-0.3207	-0.0008328	0.0006086	-0.0089543
0.9000	-0.0110	1.0431	-1.2021	-0.3001	-0.0007761	0.0007671	-0.0090967
1.0000	0.0521	1.0725	-1.1953	-0.2813	-0.0007232	0.0009181	-0.0092136
2.0000	0.5638	1.3098	-1.1[illegible]71	-0.1528	-0.0003255	0.0020486	-0.0096740
3.0000	0.9510	1.4873	-1.1995	-0.0791	-0.0000626	0.0027162	-0.0096701
4.0000	1.2712	1.6315	-1.2360	-0.0301	0.0001302	0.0031429	-0.0095581
5.0000	1.5485	1.7540	-1.2780	0.0051	0.0002796	0.0034341	-0.0094213
6.0000	1.7955	1.8609	-1.3221	0.0317	0.0003995	0.0036421	-0.0092846
7.0000	2.0198	1.9562	-1.3664	0.0525	0.0004982	0.0037959	-0.0091558
8.0000	2.2262	2.0422	-1.4104	0.0693	0.0005811	0.0039126	-0.0090371
9.0000	2.4183	2.1208	-1.4534	0.0830	0.0006518	0.0040027	-0.0089284
10.0000	2.5985	2.1933	-1.4954	0.0944	0.0007128	0.0040734	-0.0088293
100.	9.683	4.315	-3.236	0.1768	0.001404	0.004097	-0.007338
10000.	99.75	14.11	-12.89	0.08136	0.001395	0.004022	-0.007311
1000000.	999.8	44.71	-43.46	0.02756	0.001369	0.004087	-0.007275

Table II
STEADY-STATE GAINS (CONTINUATION)

q	K_{41}	K_{42}	K_{43}	K_{44}	K_{45}	K_{46}	K_{47}
0.0100	0.8078	-0.4555	1.2513	0.5877	0.0013943	-0.0019537	0.0046583
0.0200	1.0359	-0.5134	1.5640	0.9003	0.0017551	-0.0032934	0.0055174
0.0300	1.1847	-0.5372	1.7579	1.0384	0.0019819	-0.0042778	0.0059098
0.0400	1.2971	-0.5485	1.8993	1.1417	0.0021488	-0.0050622	0.0061096
0.0500	1.388[illegible]	-0.5537	2.0106	1.2246	0.0022812	-0.0057153	0.0062073
0.0600	1.4[illegible]	-0.5555	2.1024	1.2938	0.0023909	-0.0062741	0.0062436
0.0700	1.5321	-0.5552	2.1804	1.3534	0.0024847	-0.0067615	0.0062403
0.0800	1.5913	-0.5536	2.2483	1.4056	0.0025665	-0.0071926	0.0062100
0.0900	1.6446	-0.5511	2.3083	1.4521	0.0026391	-0.0075779	0.0061607
0.1000	1.6930	-0.5479	2.3621	1.4941	0.0027042	-0.0079253	0.0060979
0.2000	2.0292	-0.5062	2.7158	1.7750	0.0031346	-0.0101702	0.0052047
0.3000	2.2391	-0.4649	2.9210	1.9406	0.0033822	-0.0113171	0.0042906
0.4000	2.3932	-0.4286	3.0655	2.0574	0.0035524	-0.0119816	0.0034925
0.5000	2.5152	-0.3967	3.1770	2.1470	0.0036795	-0.0123867	0.0028113
0.6000	2.6163	-0.3685	3.2680	2.2195	0.0037793	-0.0126378	0.0022303
0.7000	2.7025	-0.3433	3.3451	2.2801	0.0038602	-0.0127916	0.0017319
0.8000	2.7778	-0.3207	3.4121	2.3321	0.0039272	-0.0128814	0.0013014
0.9000	2.8445	-0.3001	3.4716	2.3776	0.0039837	-0.0129279	0.0009270
1.0000	2.9044	-0.2813	3.5252	2.4179	0.0040319	-0.0129445	0.0005990
2.0000	3.3008	-0.1528	3.8985	2.6762	0.0042807	-0.0125765	-0.0012605
3.0000	3.5326	-0.0791	4.1497	2.8243	0.0043565	-0.0121249	-0.0020369
4.0000	3.6967	-0.0301	4.3545	2.9305	0.0043729	-0.0117733	-0.0024505
5.0000	3.8241	0.0051	4.5343	3.0148	0.0043631	-0.0115036	-0.0027053
6.0000	3.9287	0.0317	4.6980	3.0858	0.0043406	-0.0112919	-0.0028785
7.0000	4.0178	0.0525	4.8504	3.1477	0.0043118	-0.0111214	-0.0030050
8.0000	4.0956	0.0693	4.9940	3.2032	0.0042799	-0.0109808	-0.0031024
9.0000	4.1649	0.0830	5.1306	3.2536	0.0042468	-0.0108625	-0.0031808
10.0000	4.2276	0.0944	5.2614	3.3032	0.0042136	-0.0107613	-0.0032459
100.	6.003	0.1788	11.55	4.876	0.003215	-0.009091	-0.004291
10000.	15.46	0.08136	101.3	14.24	0.002500	-0.007414	-0.004244
1000000.	46.00	0.02756	1001.	44.75	0.002430	-0.007279	-0.004111

Table III
PERFORMANCE OF THE OPTIMAL CONTROL

INITIAL CONDITIONS
X1= 0.   X2= 0.
X3= 0.   X4= 0.
SUN DIRECTION= 0. DEGREES

Time	Control	Drift	Peak Drift	ΔV Cost
14.6000	4.57906E-04	4.05241E-03	4.83045E-03	6.48640E-02
14.6100	4.36649E-04	4.09946E-03	4.83045E-03	6.48684E-02
14.6200	4.15838E-04	4.14693E-03	4.83045E-03	6.48725E-02
14.6300	3.95479E-04	4.19485E-03	4.83045E-03	6.48765E-02
14.6400	3.75574E-04	4.24326E-03	4.83045E-03	6.48802E-02
14.6500	3.56128E-04	4.29221E-03	4.83045E-03	6.48838E-02
14.6600	3.37146E-04	4.34172E-03	4.83045E-03	6.48872E-02
14.6700	3.18631E-04	4.39184E-03	4.83045E-03	6.48904E-02
14.6800	3.00589E-04	4.44261E-03	4.83045E-03	6.48934E-02
14.6900	2.83022E-04	4.49407E-03	4.83045E-03	6.48962E-02
14.7000	2.65935E-04	4.54627E-03	4.83045E-03	6.48989E-02
14.7100	2.49332E-04	4.59924E-03	4.83045E-03	6.49013E-02
14.7200	2.33217E-04	4.65304E-03	4.83045E-03	6.49037E-02
14.7300	2.17595E-04	4.70770E-03	4.83045E-03	6.49059E-02
14.7400	2.02470E-04	4.76326E-03	4.83045E-03	6.49079E-02
14.7500	1.87845E-04	4.81977E-03	4.83045E-03	6.49098E-02
14.7600	1.73726E-04	4.87727E-03	4.87727E-03	6.49115E-02
14.7700	1.60117E-04	4.93580E-03	4.93580E-03	6.49131E-02
14.7800	1.47023E-04	4.99540E-03	4.99540E-03	6.49146E-02
14.7900	1.34447E-04	5.05612E-03	5.05612E-03	6.49159E-02
14.8000	1.22395E-04	5.11799E-03	5.11799E-03	6.49171E-02
14.8100	1.10871E-04	5.18105E-03	5.18105E-03	6.49182E-02
14.8200	9.98815E-05	5.24534E-03	5.24534E-03	6.49192E-02
14.8300	8.94302E-05	5.31090E-03	5.31090E-03	6.49201E-02
14.8400	7.95229E-05	5.37777E-03	5.37777E-03	6.49209E-02
14.8500	7.01648E-05	5.44597E-03	5.44597E-03	6.49216E-02
14.8600	6.13617E-05	5.51555E-03	5.51555E-03	6.49223E-02
14.8700	5.31193E-05	5.58654E-03	5.58654E-03	6.49228E-02
14.8800	4.54436E-05	5.65896E-03	5.65896E-03	6.49232E-02
14.8900	3.83407E-05	5.73285E-03	5.73285E-03	6.49236E-02
14.9000	3.18170E-05	5.80824E-03	5.80824E-03	6.49239E-02
14.9100	2.58791E-05	5.88515E-03	5.88515E-03	6.49242E-02
14.9200	2.05338E-05	5.96360E-03	5.96360E-03	6.49244E-02
14.9300	1.57882E-05	6.04362E-03	6.04362E-03	6.49246E-02
14.9400	1.16494E-05	6.12524E-03	6.12524E-03	6.49247E-02
14.9500	8.12499E-06	6.20846E-03	6.20846E-03	6.49248E-02
14.9600	5.22278E-06	6.29331E-03	6.29331E-03	6.49248E-02
14.9700	2.95076E-06	6.37981E-03	6.37981E-03	6.49248E-02
14.9800	1.31722E-06	6.46796E-03	6.46796E-03	6.49249E-02
14.9900	3.30731E-07	6.55778E-03	6.55778E-03	6.49249E-02
15.0000	0.	6.64929E-03	6.64929E-03	6.49249E-02

Table IV
PERFORMANCE OF THE MODIFIED CONTROL

INITIAL CONDITIONS
X1= 0. X2= 0.
X3= 0. X4= 0.
SUN DIRECTION= 0. DEGREES

Time	Control	Drift	Peak Drift	ΔV Cost
14.6000	4.30926E-03	3.78023E-03	4.83928E-03	6.77860E-02
14.6100	4.34131E-03	3.81430E-03	4.83928E-03	6.78294E-02
14.6200	4.37299E-03	3.84810E-03	4.83928E-03	6.78731E-02
14.6300	4.40426E-03	3.88158E-03	4.83928E-03	6.79171E-02
14.6400	4.43511E-03	3.91473E-03	4.83928E-03	6.79615E-02
14.6500	4.46549E-03	3.94752E-03	4.83928E-03	6.80061E-02
14.6600	4.49539E-03	3.97993E-03	4.83928E-03	6.80511E-02
14.6700	4.52477E-03	4.01192E-03	4.83928E-03	6.80963E-02
14.6800	4.55360E-03	4.04349E-03	4.83928E-03	6.81419E-02
14.6900	4.58187E-03	4.07460E-03	4.83928E-03	6.81877E-02
14.7000	4.60954E-03	4.10524E-03	4.83928E-03	6.82338E-02
14.7100	4.63560E-03	4.13539E-03	4.83928E-03	6.82802E-02
14.7200	4.66302E-03	4.16503E-03	4.83928E-03	6.83268E-02
14.7300	4.68878E-03	4.19414E-03	4.83928E-03	6.83737E-02
14.7400	4.71386E-03	4.22270E-03	4.83928E-03	6.84208E-02
14.7500	4.73824E-03	4.25070E-03	4.83928E-03	6.84682E-02
14.7600	4.76190E-03	4.27813E-03	4.83928E-03	6.85158E-02
14.7700	4.78483E-03	4.30496E-03	4.83928E-03	6.85637E-02
14.7800	4.80700E-03	4.33119E-03	4.83928E-03	6.86117E-02
14.7900	4.82840E-03	4.35680E-03	4.83928E-03	6.86600E-02
14.8000	4.84901E-03	4.38177E-03	4.83928E-03	6.87085E-02
14.8100	4.86883E-03	4.40611E-03	4.83928E-03	6.87572E-02
14.8200	4.88783E-03	4.42979E-03	4.83928E-03	6.88061E-02
14.8300	4.90601E-03	4.45281E-03	4.83928E-03	6.88551E-02
14.8400	4.92335E-03	4.47516E-03	4.83928E-03	6.89044E-02
14.8500	4.93984E-03	4.49682E-03	4.83928E-03	6.89538E-02
14.8600	4.95548E-03	4.51780E-03	4.83928E-03	6.90033E-02
14.8700	4.97024E-03	4.53808E-03	4.83928E-03	6.90530E-02
14.8800	4.98413E-03	4.55766E-03	4.83928E-03	6.91029E-02
14.8900	4.99713E-03	4.57653E-03	4.83928E-03	6.91528E-02
14.9000	5.00924E-03	4.59469E-03	4.83928E-03	6.92029E-02
14.9100	5.02045E-03	4.61213E-03	4.83928E-03	6.92531E-02
14.9200	5.03075E-03	4.62884E-03	4.83928E-03	6.93034E-02
14.9300	5.04015E-03	4.64484E-03	4.83928E-03	6.93538E-02
14.9400	5.04863E-03	4.66010E-03	4.83928E-03	6.94043E-02
14.9500	5.05620E-03	4.67463E-03	4.83928E-03	6.94549E-02
14.9600	5.06285E-03	4.68844E-03	4.83928E-03	6.95055E-02
14.9700	5.06857E-03	4.70151E-03	4.83928E-03	6.95562E-02
14.9800	5.07338E-03	4.71385E-03	4.83928E-03	6.96069E-02
14.9900	5.07726E-03	4.72545E-03	4.83928E-03	6.96577E-02
15.0000	5.08022E-03	4.73633E-03	4.83928E-03	6.97085E-02

remains constant at 0.00483 until the last 0.25 time units of the mission, then increases to a value of 0.00665 at the terminal time of 15. With the modified control, peak drift never exceeds 0.00484 during the entire mission. Cost (ΔV) is about 7% greater (0.0697 verses 0.0649) than that of the optimal control system but peak drift is smaller by more than 25% (0.00484 verses 0.00665). For longer missions, the difference in cost would be even less, while the reduction in peak drift would be exactly the same. The comparison is summarized in Table V.

Table V
COMPARISON - OPTIMAL VS MODIFIED CONTROL (T=15)

	Optimal Control	Modified Control
$\int$ (drift squared) dt	0.0002153	0.0002304
$\int$ (control squared) dt	0.0003088	0.0003370
Total Quadratic Cost	0.0005241	0.0005674
Peak Drift	0.006649	0.004839
ΔV Cost	0.06492	0.06971

Performance calculations on the modified control system were made using a number of q values which range from 0.01 to 10. The computer program used is described in Appendix B.

As a result of two important findings, the performance of the modified control system can be summarized in an astonishingly simple graph which is shown in Figure 5. The

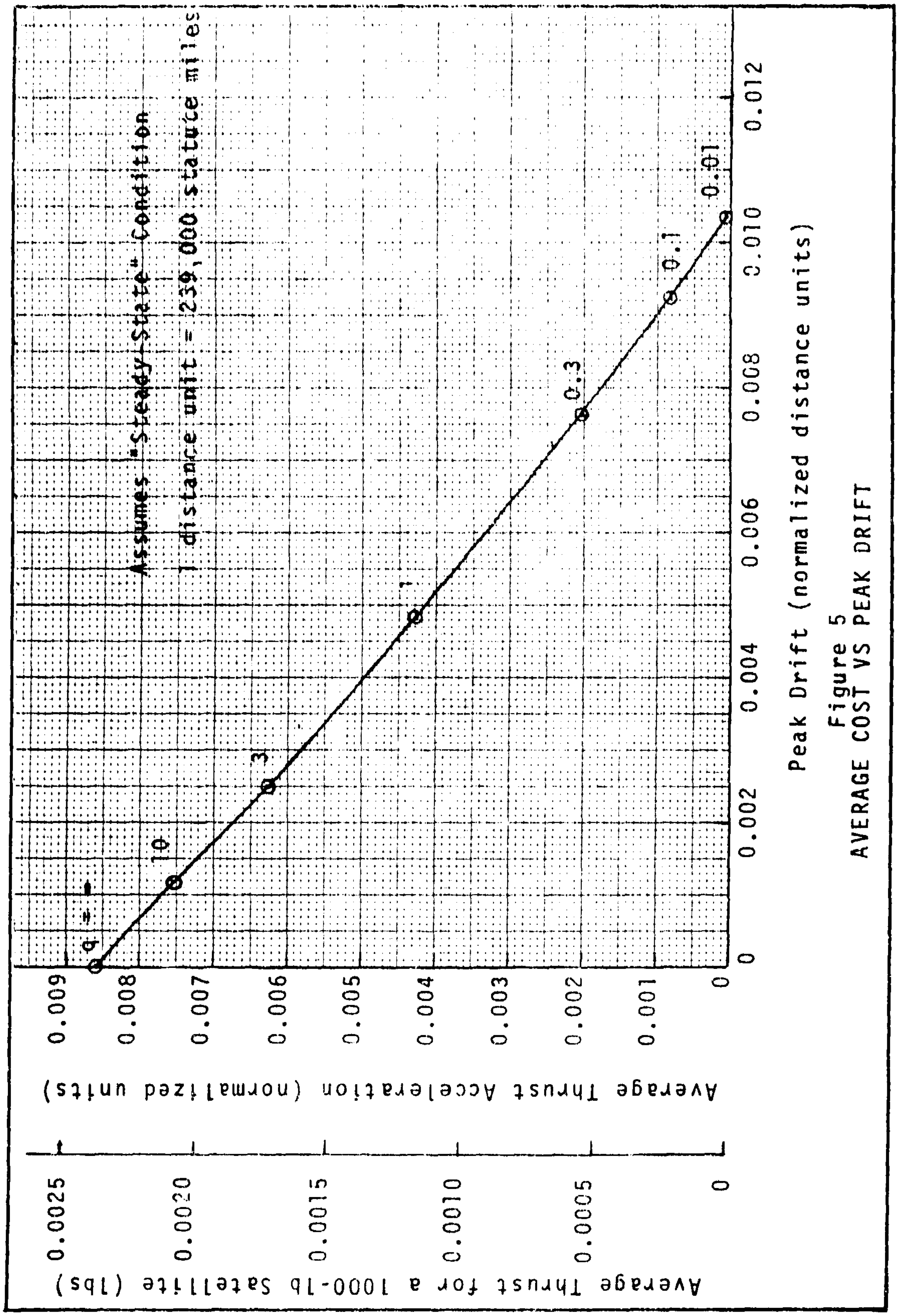


Figure 5
AVERAGE COST VS PEAK DRIFT

findings which permit this type of display are that

(1) After a brief transitory period, the peak drift is uniquely determined by q (neither mission duration nor initial conditions affect the drift after the transient).

(2) After the transitory period, the control becomes periodic with a magnitude which is roughly constant (the mean value of this control magnitude therefore can be used to represent the system cost in terms of cost per unit time).

The graph of Figure 5 essentially answers the question posed earlier (see page 7) of whether or not it would be less costly to thrust, keeping the satellite on station perfectly rather than permitting a finite drift. As seen by the graph, cost decreases, almost linearly, with permissible drift. Perfect station keeping is decidedly a more costly option.

Further study of the modified control system was accomplished with the aid of an analog computer. This additional study is described in the next chapter along with more details of the performance traits noted in this chapter.

IV. Analog Simulation

The system dynamics and the modified control system were simulated on an Electronic Associates, Inc. "PACE TR48" analog computer. This machine has a repetitive operation feature and a built-in oscilloscope which, when used together, provides a near instantaneous display of the system response. Recordings were made on an "X-Y" plotter of the most illustrative and significant runs (with the computer in normal operation mode).

The control is generated by 14 constant gain feedback paths. Two control systems were wired and tested, one having gains appropriate for a q of 0.01 and the other for a q of 1.0 (see Tables I and II for th gain values). The circuitry used for the simulation is described in Appendix C.

Several important aspects of the system are discovered which would have been difficult to detect from digital data. Discoveries were made concerning the steady-state tendency, the effect of initial conditions, and the existence of a stable orbit without control.

Steady-State Tendency

Regardless of initial conditions or the value of q, the control system always maneuvers the satellite into a cyclic, steady-state trajectory about the L4 point. The period of rotation about the L4 point in this state is always precisely half that of the sun (π/w time units). This steady-state tendency is illustrated in Figure 6. The

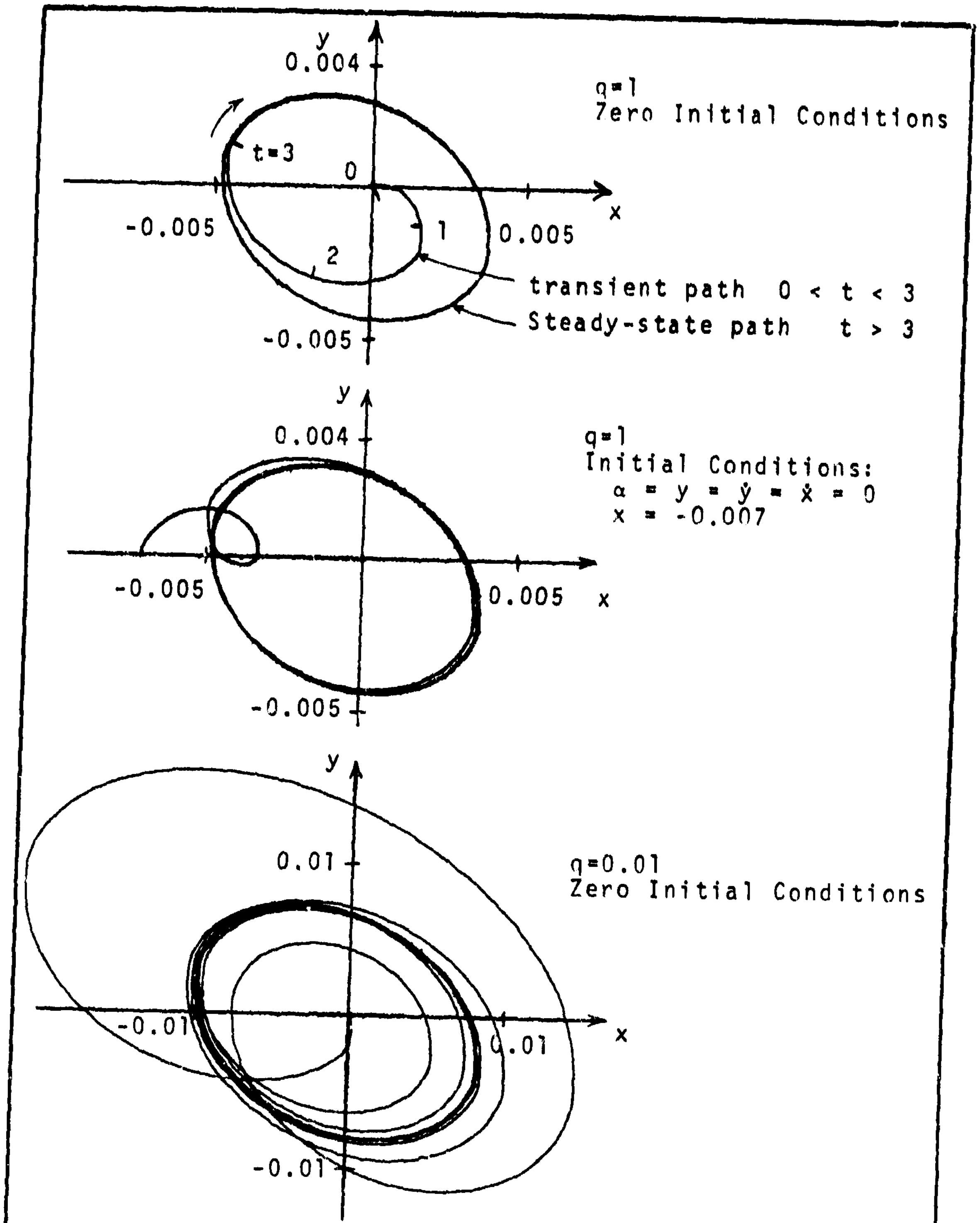


Figure 6

TRAJECTORIES WITH CONTROL

steady-state trajectory resembles an off-center ellipse. Its size is a function only of the value of q. For comparison purposes, the trajectory for a q of zero (no control) is illustrated in Figure 7. Once the satellite reaches the steady-state path, it remains there for as long as the control system is in operation.

Although initial conditions have no effect on the steady-state trajectory, they do, however, affect the initial path of the satellite. Before reaching the steady-state condition, depending upon initial conditions, the satellite may drift out even beyond the peak drift of the steady-state trajectory as in Figure 6. During the transient, the control system is apparently manuevering the satellite to place it on the steady-state path with the proper velocity and a particular phase relationship relative to the time of month. Initial conditions can always be chosen, of course, which would place the satellite precisely on the steady-state trajectory initially, therefore eliminating the transient path entirely. For a given time of month, i.e., initial sun direction α, such a set of initial conditions are obviously unique.

Initial Conditions for Minimum Cost

Once the steady-state conditions are established, the control becomes cyclic much the same as does the satellite trajectory. The control is plotted as a vector (U_y verses U_x with time as a parameter) in Figure 8 for a q of 1.0 and with zero initial conditions. During the steady-state hase, the control magnitude is roughly constant and there-

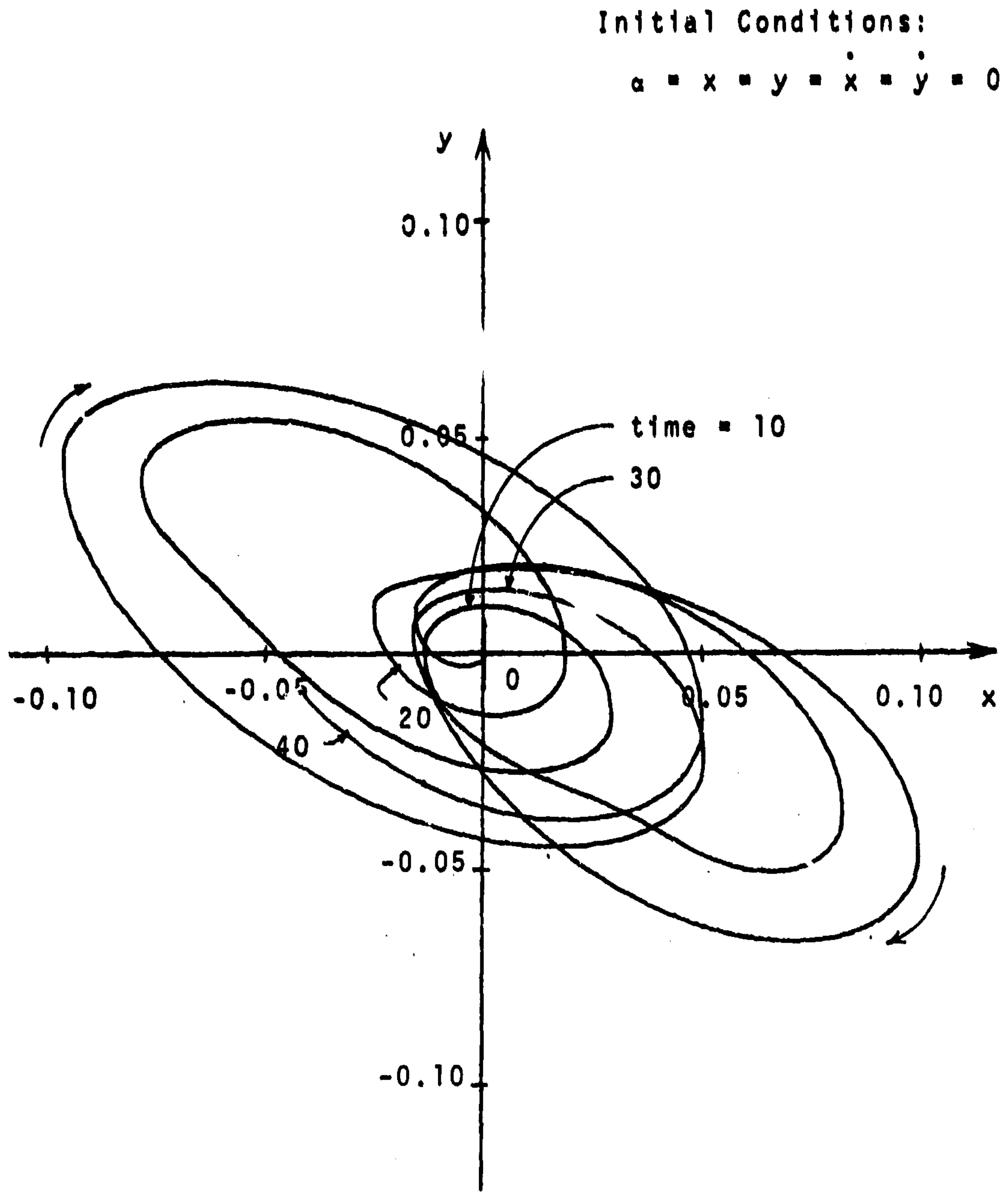


Figure 7

TRAJECTORY WITHOUT CONTROL

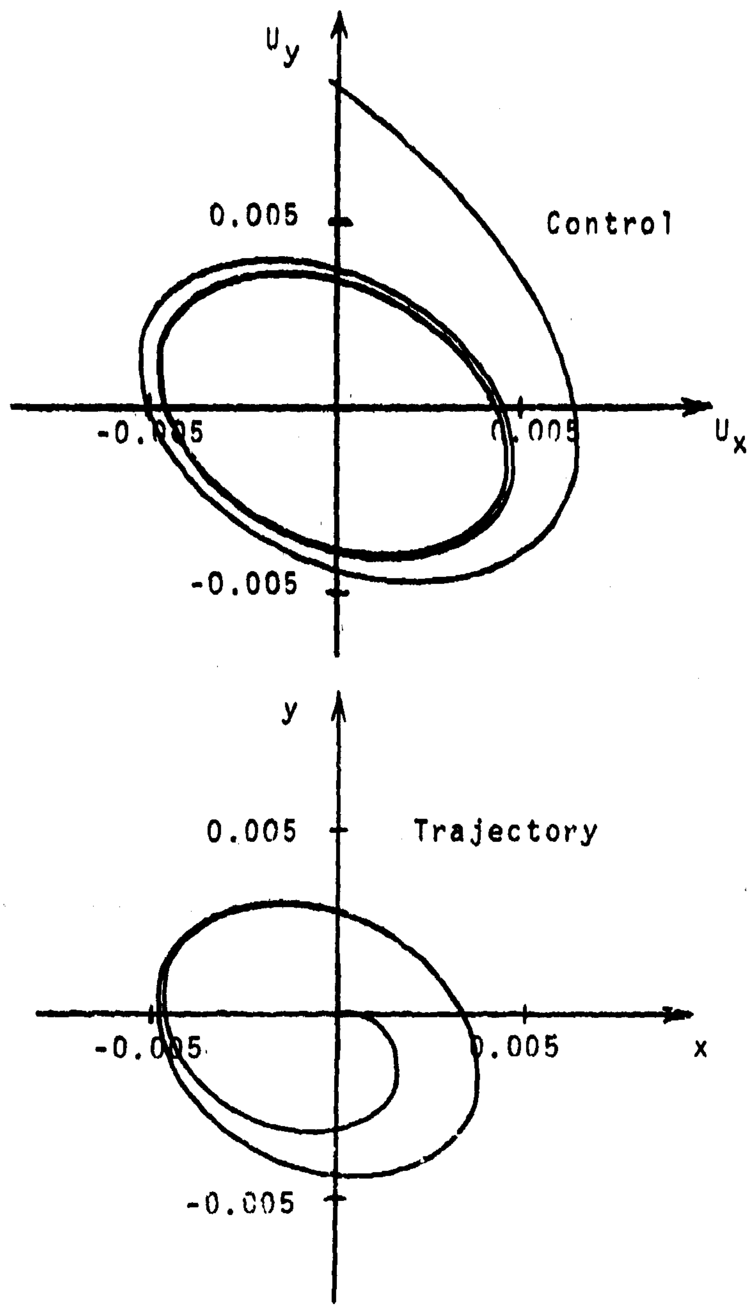


Figure 8

RESPONSE WITH ZERO I.C., q=1

fore, the system cost increases linearly with time during this phase. The magnitude of the control during the steady-state phase is unaffected by initial conditions, however this is not so during the transient phase. The magnitude of the control during the transient phase, and therefore the system cost, can be minimized by a proper choice of initial conditions. Such a set and its effect on the control is illustrated in Figure 9. Although the control can be minimized, it is done so at the sacrifice of permitting the satellite to drift beyond the steady-state path. A compromise is possible however, as shown in Figure 10. In this situation the satellite remains completely inside the steady-state path but the reduction in the control magnitude is not quite as great as before. For a q of 0.01, no better initial conditions are found than those which place the satellite on the steady-state trajectory initially (see Figures 11 and 12).

No matter how much the control is minimized during the transient phase, the effect on the system cost will be small if long duration missions are considered. In Figure 13 the system cost with a minimized control magnitude during the transient period is compared with the system cost in which there is no transient period. Certainly the saving is insignificant for missions of more than a few months.

Optimum Initial Conditions Without Control

Extrapolation of the "cost verses peak drift" curve of Figure 5 suggests that a stable orbit having a peak drift of

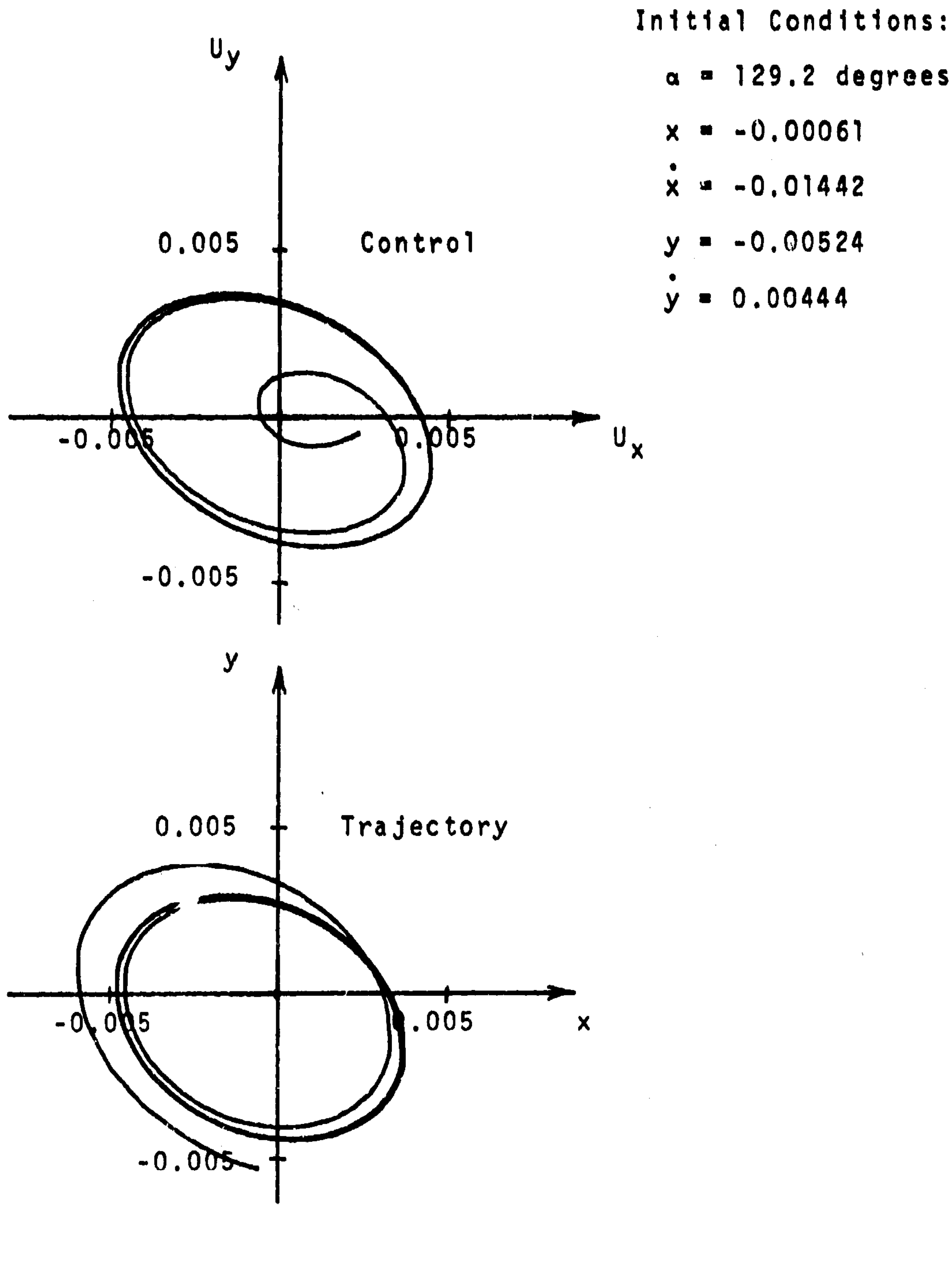


Figure 9

RESPONSE WITH "MINIMUM COST" I.C., q=1

Initial Conditions:

α = 138.3 degrees

x = 0

$\dot{x}$ = -0.01134

y = -0.00442

$\dot{y}$ = 0.00496

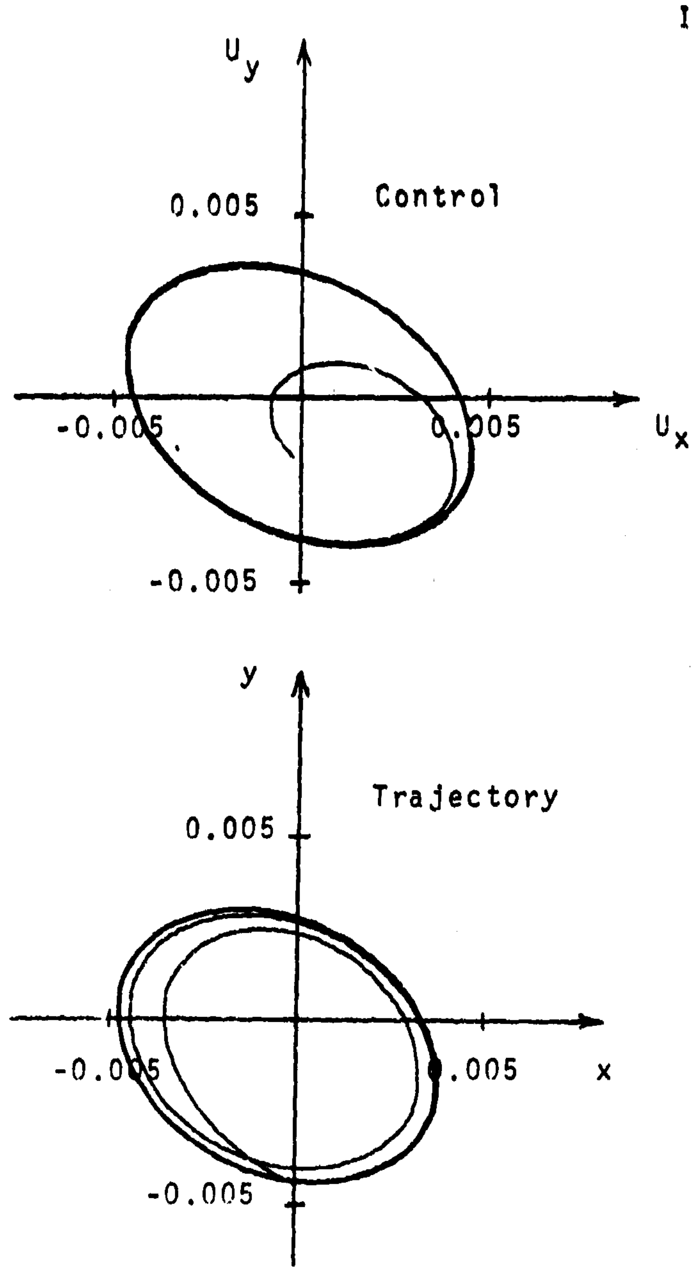


Figure 10

RESPONSE WITH "COMPROMISE" I.C., q=1

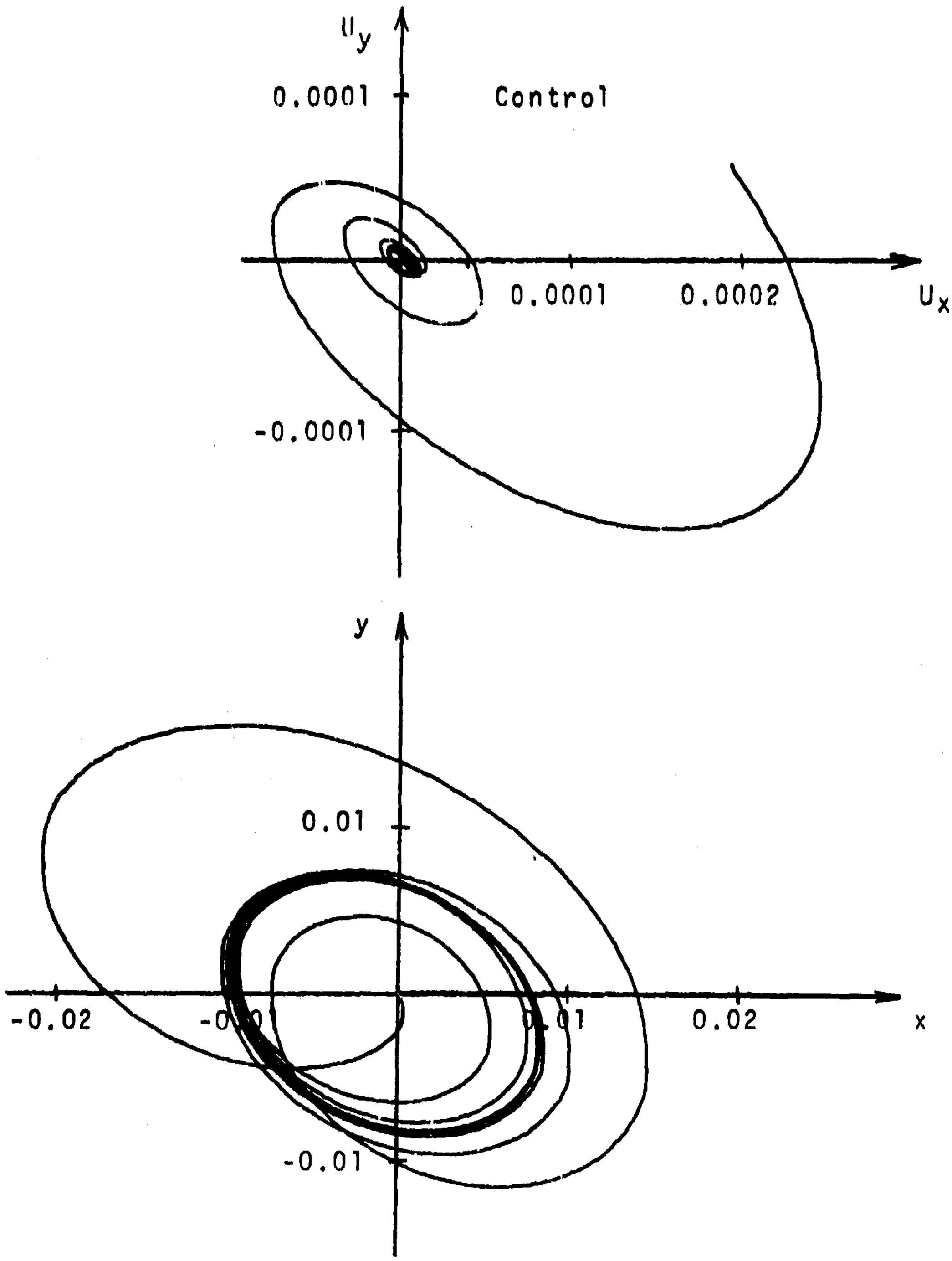
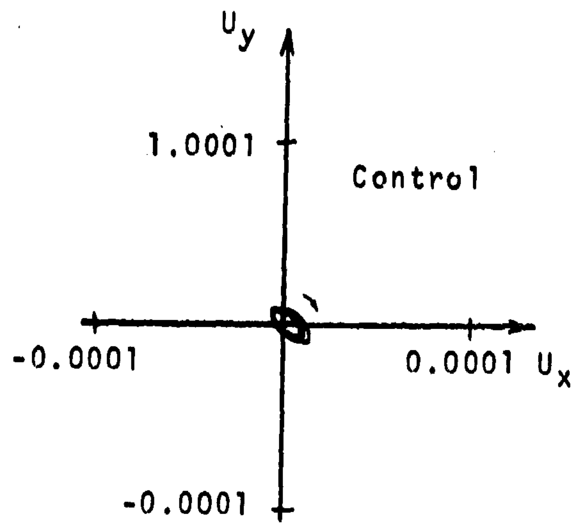


Figure 11

RESPONSE WITH ZERO I.C., q=0.01



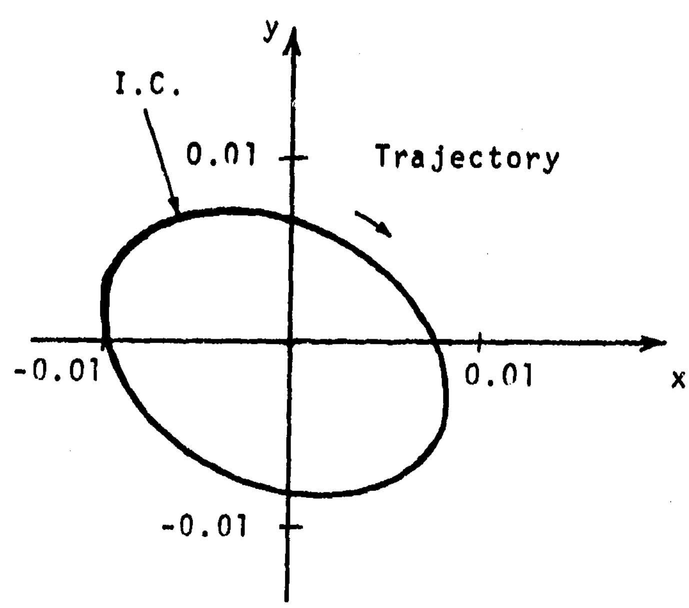


Figure 12

RESPONSE WITH "NO-TRANSIENT" I.C., q=0.01

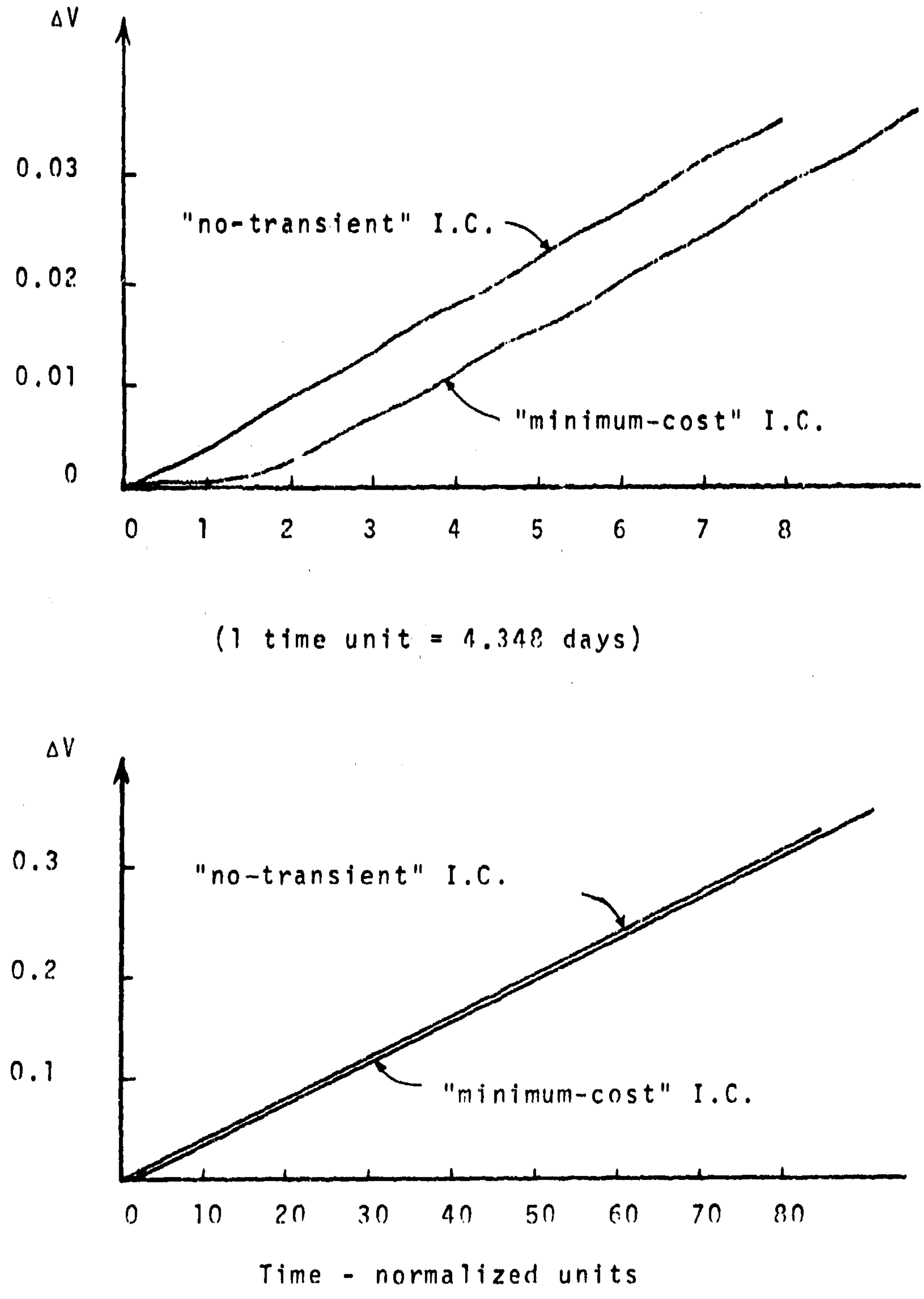


Figure 13

TOTAL COST VS MISSION DURATION, q=1

about 0.011 would exist if q were zero, that is to say without a control system at all! Further investigation shows that, for any value of q, the controlled system under steady-state conditions always exhibits the same phase relationships between sun direction, satellite position and velocity (see Figures 14 thru 17). This fact implies that perhaps a particular phase relationship is mandatory for the establishment of stable orbits, and in particular for a stable orbit with a q of zero (no control). Indeed this turns out to be the case. If the same phase relations are established by appropriate selection of the initial conditions, the satellite does in fact remain in a stable orbit without a control system and furthermore one that has a peak drift of about 0.011. This is not inconsistent with the findings of Dr. Lynn E. Wolver (Ref 5) and Capt. Paul Ulshafer (Ref 4) which show that a fairly stable orbit can be achieved having a peak drift on only 0.013 during a one-year mission provided a judicious choice of initial conditions is made.

The appropriate initial conditions can be derived from data of systems with small q values. The required data are the values of X_1, X_2, X_3, X_4, and X_7 as functions of time after reaching steady-state conditions. A time plot of these variables in steady-state conditions is shown in Figures 18 and 19 for a q of 0.01. The optimum initial conditions for any given α are those which correspond to an X_7 having the value $\sin 2\alpha$ since

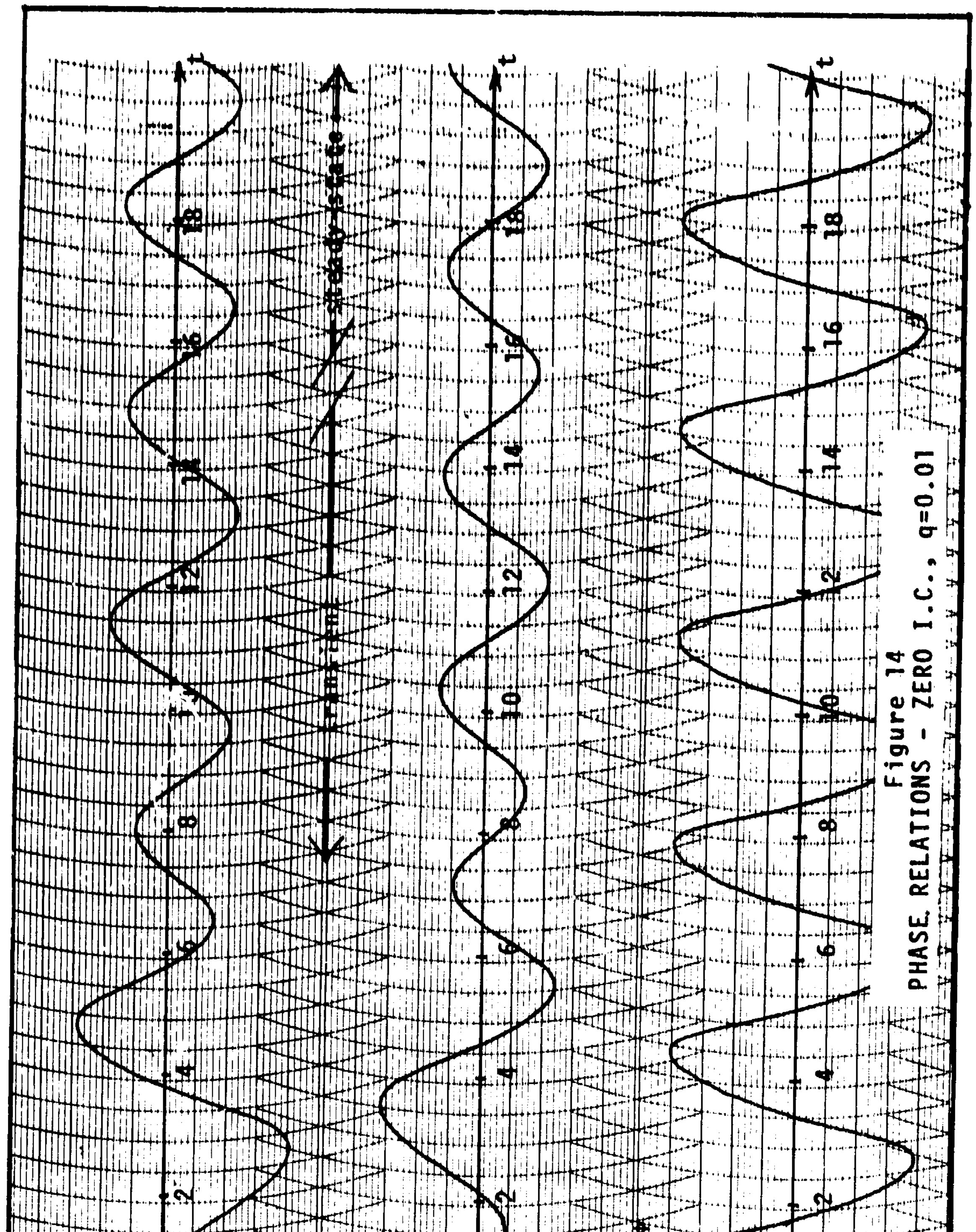


Figure 14
PHASE RELATIONS - ZERO I.C., q=0.01

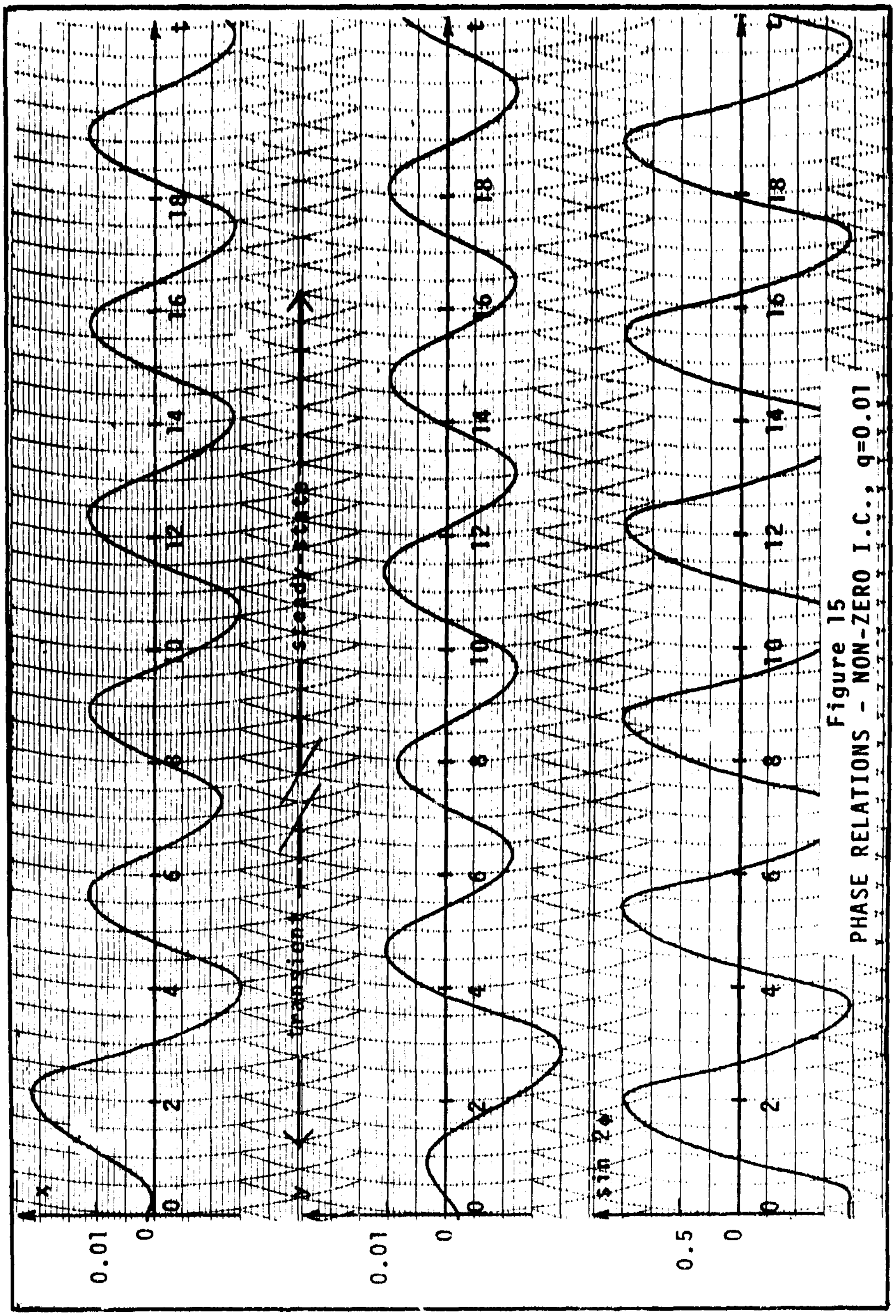


Figure 15
PHASE RELATIONS - NON-ZERO I.C., q=0.01

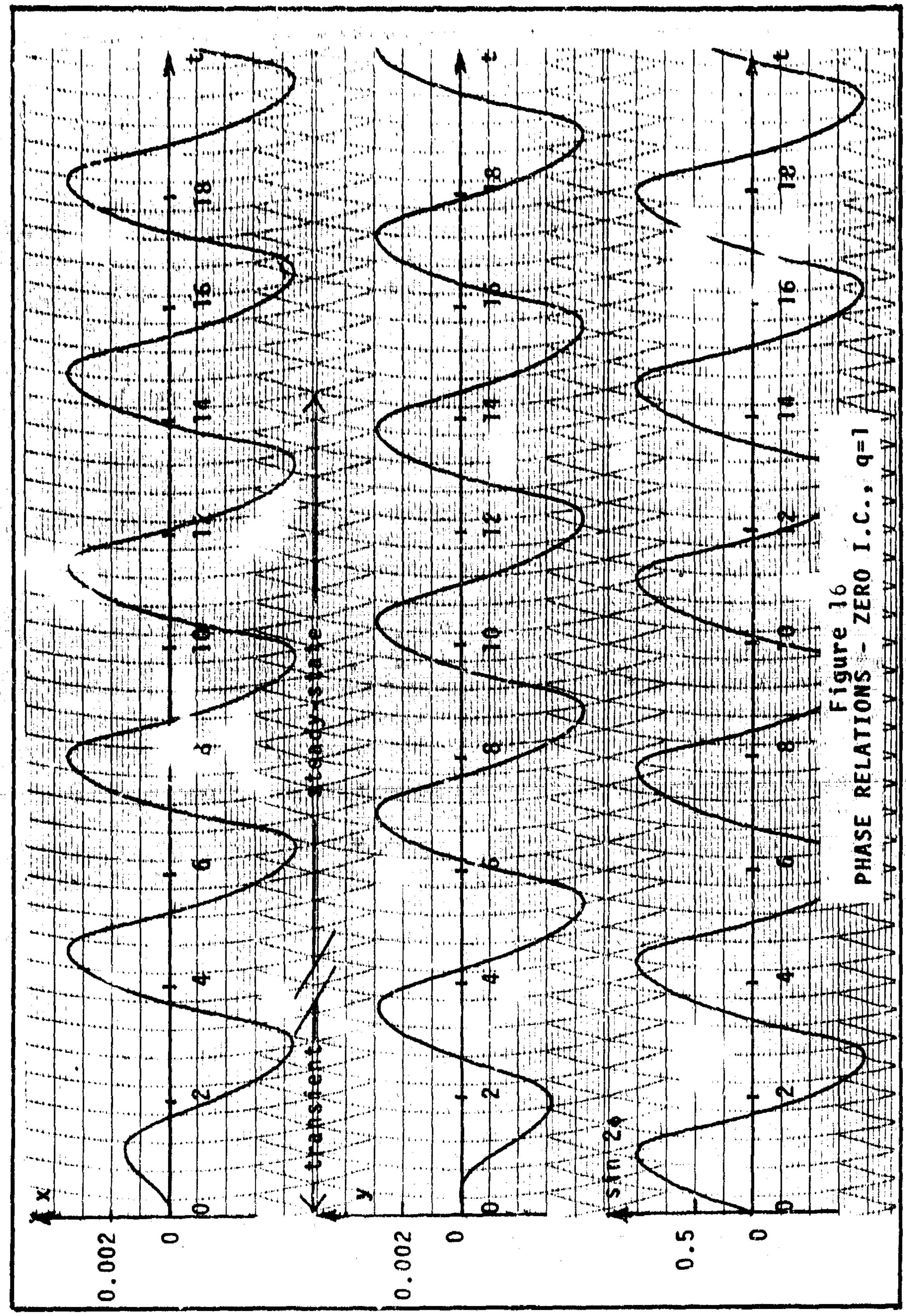


Figure 16
PHASE RELATIONS - ZERO I.C., q=1

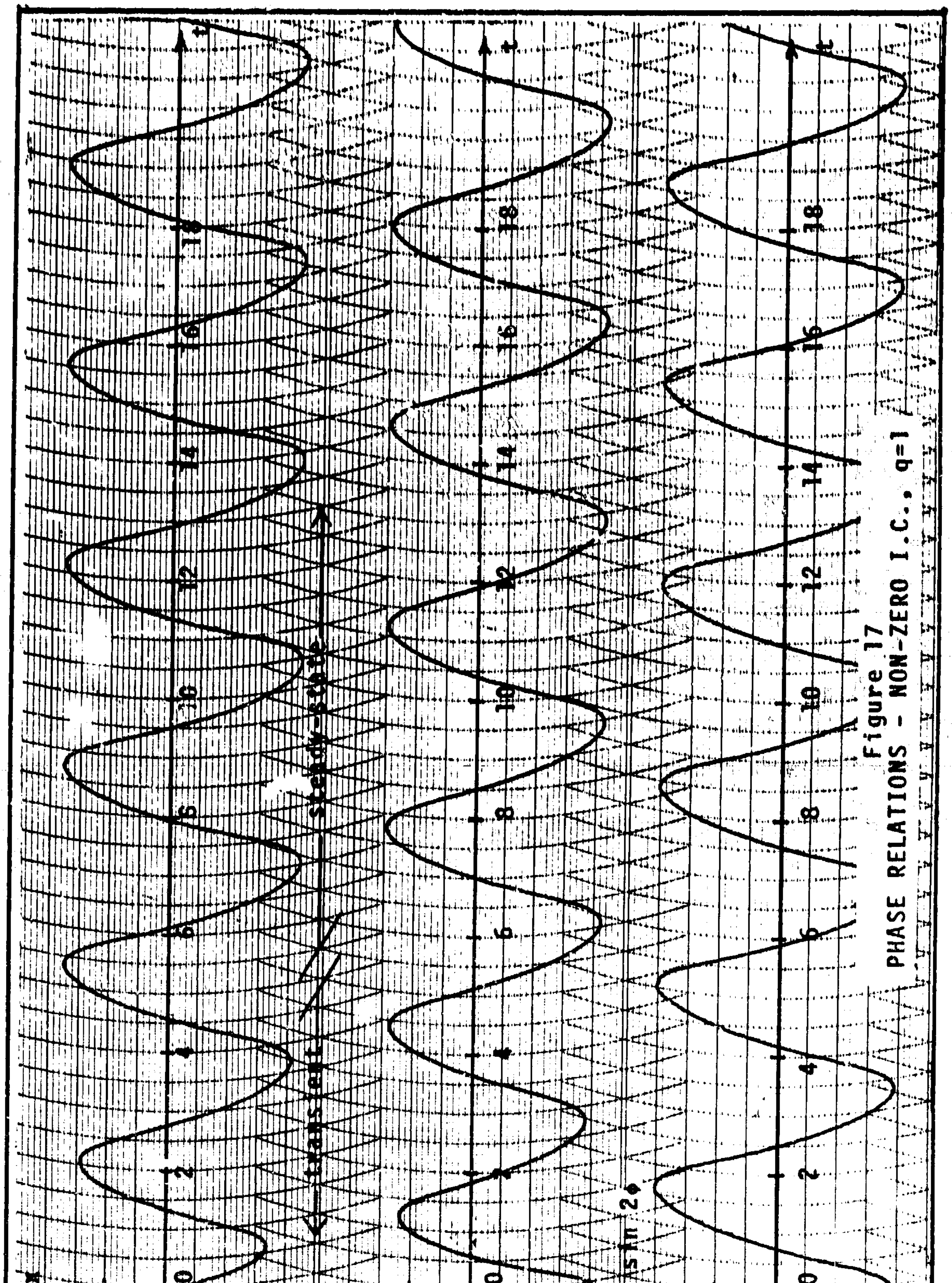


Figure 17
PHASE RELATIONS - NON-ZERO I.C., q=1

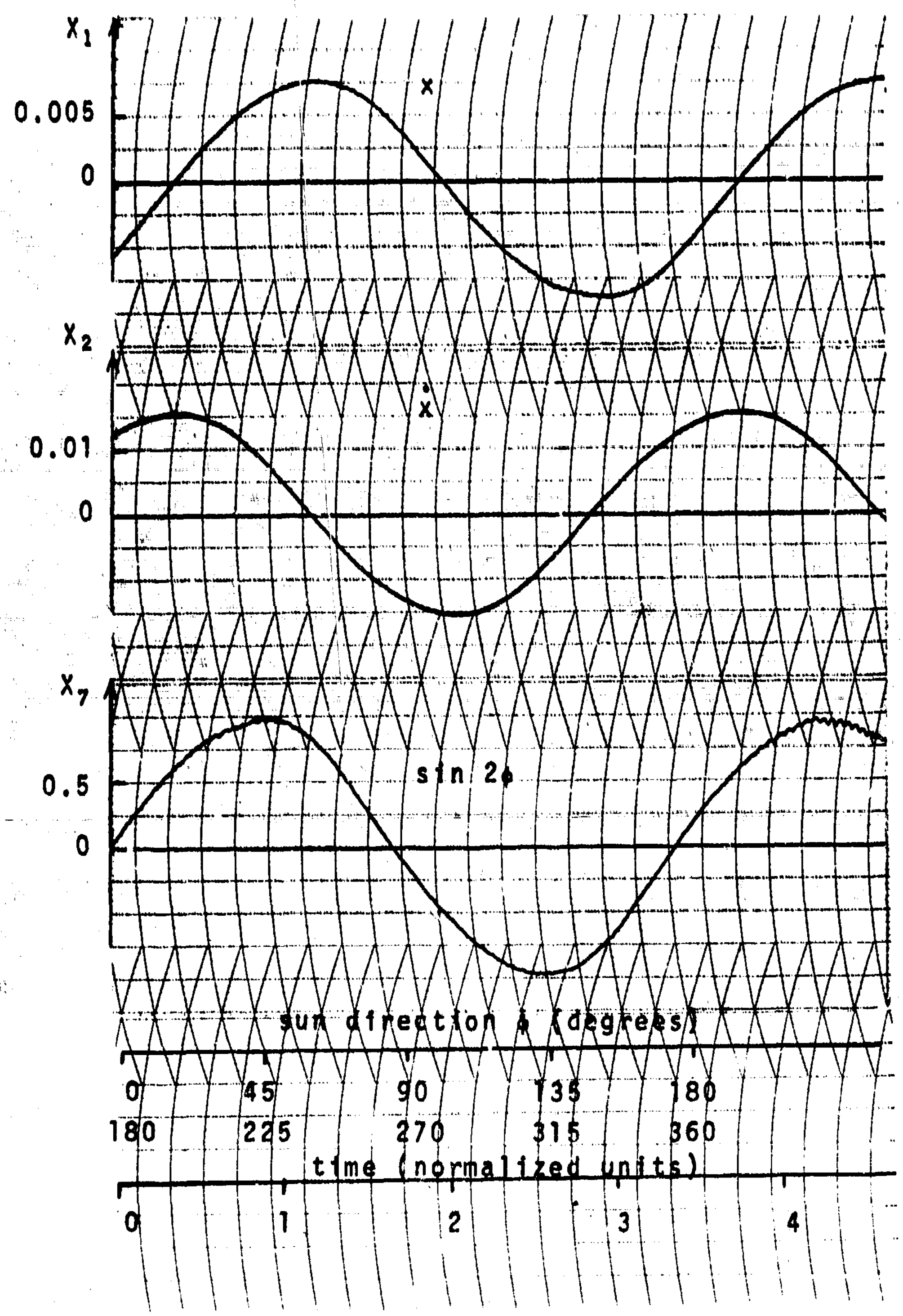


Figure 18

"STEADY-STATE" X_1 AND X_2, q=0.01

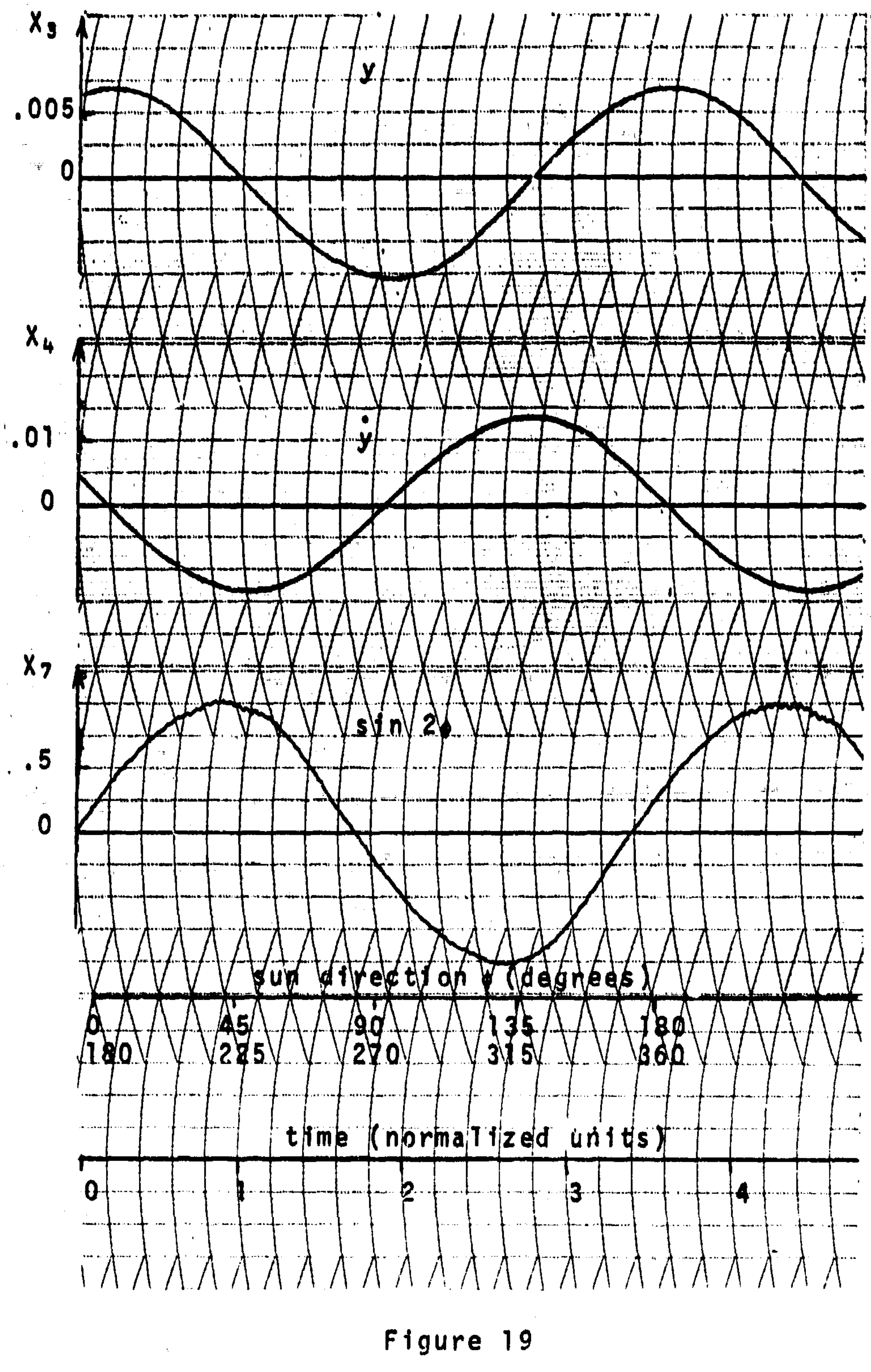


Figure 19

"STEADY STATE" X_3 AND X_4, q=0.01

$$X_7 = \sin 2 (w t + \alpha) \tag{27}$$

Fair accuracy would be obtained using the data for a system with a q of 0.01, but for a more accurate determination, the data from Tables VI (q = 0.01), VII (q = 0.02) and VIII (q = 0.03) can be extrapolated to give values which would correspond to a q of zero.

The analog plot of Figure 20 illustrates the stable orbit without control. For this plot, the extrapolation method just described was used to determine the initial conditions. The initial sun direction α was chosen such that X_7 corresponded to an exact table entry (7.2787599 E-03) but any α could have been chosen. The orbit remains reasonably stable for nearly three years according to the analog plot. The slight instability is due only to accuracy limitations of the analog computer. The potentiometer settings become extremely critical for uncontrolled orbits of long duration such as the one depicted here.

Table VI
DIGITAL DATA - "STEADY-STATE," q = 0.01

time	X_7	X_1	X_2	X_3	X_4
78.10	-4.3896002E-04	-6.3817589E-03	1.3680399E-02	6.7300786E-03	5.0151411E-03
78.20	1.8354077E-01	-4.9262415E-03	1.5348984E-02	7.1035596E-03	2.4337892E-03
78.30	3.6125495E-01	-3.3296573E-03	1.6493918E-02	7.2140419E-03	-2.3021712E-04
78.40	5.2663691E-01	-1.6464683E-03	1.7076291E-02	7.0577898E-03	-2.8860854E-03
78.50	6.7404101E-01	6.5921806E-05	1.7076331E-02	6.6401561E-03	-5.4433390E-03
78.60	7.9843644E-01	1.7491192E-03	1.6494061E-02	5.9753969E-03	-7.8148802E-03
78.70	8.9557413E-01	3.3457249E-03	1.5349300E-02	5.0861847E-03	-9.9199358E-03
78.80	9.6213949E-01	4.8012858E-03	1.3680991E-02	4.0028378E-03	-1.1686791E-02
78.90	9.9586017E-01	6.0661473E-03	1.1545888E-02	2.7622902E-03	-1.3055217E-02
79.00	9.9558487E-01	7.0971431E-03	9.0166515E-03	1.4068367E-03	-1.3978513E-02
79.10	9.6132299E-01	7.8590671E-03	6.1793920E-03	-1.7302816E-05	-1.4425103E-02
79.20	8.9424430E-01	8.3258770E-03	3.1307665E-03	-1.4615547E-03	-1.4379618E-02
79.30	7.9663869E-01	8.4815874E-03	-2.5296395E-05	-2.8766435E-03	-1.3843438E-02
79.40	6.7183811E-01	8.3208236E-03	-3.1811245E-03	-4.2142710E-03	-1.2834670E-02
79.50	5.2410129E-01	7.8490099E-03	-6.2289684E-03	-5.4287632E-03	-1.1387556E-02
79.60	3.5647457E-01	7.0821969E-03	-9.0646825E-03	-6.4786315E-03	-9.5513266E-03
79.70	1.8061056E-01	6.0465180E-03	-1.1591293E-02	-7.3279923E-03	-7.3885498E-03
79.80	-3.4189717E-03	4.7773049E-03	-1.3722334E-02	-7.9477982E-03	-4.9730062E-03
79.90	-1.8733180E-01	3.3178843E-03	-1.5384824E-02	-8.3168373E-03	-2.3871781E-03
80.00	-3.6485142E-01	1.7180987E-03	-1.6521791E-02	-8.4224635E-03	2.8056797E-04
80.10	-5.2991420E-01	3.2602516E-05	-1.7094248E-02	-8.2610358E-03	2.9390118E-03
80.20	-6.7688723E-01	-1.6810102E-03	-1.7082548E-02	-7.8380465E-03	5.4972130E-03
80.30	-8.0075327E-01	-3.3641789E-03	-1.6487067E-02	-7.1679371E-03	7.8676418E-03
80.40	-8.9728475E-01	-4.9593840E-03	-1.5328193E-02	-6.2736033E-03	9.9691932E-03
80.50	-9.6318434E-01	-6.4121191E-03	-1.3645627E-02	-5.1856129E-03	1.1729982E-02
80.60	-9.9620360E-01	-7.6727583E-03	-1.1497001E-02	-3.9411555E-03	1.3089815E-02
80.70	-9.9521533E-01	-8.6982541E-03	-8.9558940E-03	-2.5627679E-03	1.4002257E-02
80.80	-9.6025327E-01	-9.4536104E-03	-6.1092801E-03	-1.1568730E-03	1.4436221E-02
80.90	-8.9251006E-01	-9.9130740E-03	-3.0545340E-03	2.8781056E-04	1.4377016E-02
81.00	-7.9429965E-01	-1.0061010E-02	1.0392178E-04	1.7019382E-03	1.3826836E-02
81.10	-6.6897414E-01	-9.8924286E-03	3.2582006E-03	3.0372269E-03	1.2804660E-02
81.20	-5.2081175E-01	-9.4131469E-03	6.3006444E-03	4.2481050E-03	1.1345579E-02
81.30	-3.5486860E-01	-8.6395830E-03	9.1274940E-03	5.2932659E-03	9.4995673E-03
81.40	-1.7681286E-01	-7.5981869E-03	1.1642420E-02	6.1370749E-03	7.3297596E-03
81.50	7.2787599E-03	-6.3245325E-03	1.3759785E-02	6.7507790E-03	4.9102761E-03

Table VII
DIGITAL DATA - "STEADY-STATE," q = 0.02

time	X_7	X_1	X_2	X_3	X_4
78.10	-4.3896002E-04	-6.2734410E-03	1.3529430E-02	6.6216303E-03	4.9666137E-03
78.20	1.8354077E-01	-4.8339823E-03	1.5179676E-02	6.9916593E-03	2.4134664E-03
78.30	3.6125495E-01	-3.2550053E-03	1.6312066E-02	7.1015517E-03	-2.2166764E-04
78.40	5.2663691E-01	-1.5903693E-03	1.6868114E-02	6.9475888E-03	-2.8489856E-03
78.50	6.7404101E-01	1.0315560E-04	1.6886254E-02	6.5350422E-03	-5.3789852E-03
78.60	7.9843644E-01	1.7678195E-03	1.6312497E-02	5.8779915E-03	-7.7254953E-03
78.70	8.9557413E-01	3.3468549E-03	1.5180433E-02	4.9988440E-03	-9.8085899E-03
78.80	9.6213949E-01	4.7864068E-03	1.3530564E-02	3.9275730E-03	-1.1557293E-02
78.90	9.9586017E-01	6.0373648E-03	1.1419013E-02	2.7006998E-03	-1.2911985E-02
79.00	9.9558487E-01	7.0570347E-03	8.9176350E-03	1.3600533E-03	-1.3826423E-02
79.10	9.6132299E-01	7.8105951E-03	6.1115947E-03	-4.8650087E-05	-1.4269325E-02
79.20	8.9424430E-01	8.2722869E-03	3.0964890E-03	-1.4773612E-03	-1.4225440E-02
79.30	7.9663869E-01	8.4262991E-03	-2.4889758E-05	-2.8773322E-03	-1.3696094E-02
79.40	6.7183811E-01	8.2673142E-03	-3.1460446E-03	-4.2007779E-03	-1.2699162E-02
79.50	5.2410129E-01	7.8006974E-03	-6.1604011E-03	-5.4025058E-03	-1.1268486E-02
79.60	3.5847457E-01	7.0423228E-03	-8.9649503E-03	-6.4414614E-03	-9.4527465E-03
79.70	1.8061056E-01	6.0180377E-03	-1.1463778E-02	-7.2821327E-03	-7.3138222E-03
79.80	-3.4189717E-03	4.7627875E-03	-1.3571365E-02	-7.8957694E-03	-4.9246902E-03
79.90	-1.8733180E-01	3.3194244E-03	-1.5215532E-02	-8.2613709E-03	-2.3669418E-03
80.00	-3.6485142E-01	1.7372449E-03	-1.6339939E-02	-8.3664114E-03	2.7200787E-04
80.10	-5.2991420E-01	7.0303579E-05	-1.6906034E-02	-8.2072725E-03	2.9019173E-03
80.20	-6.7688723E-01	-1.6244383E-03	-1.6894396E-02	-7.7893716E-03	5.4328206E-03
80.30	-8.0075327E-01	-3.2890646E-03	-1.6305407E-02	-7.1269789E-03	7.7781240E-03
80.40	-8.9728475E-01	-4.8666900E-03	-1.5159242E-02	-6.2427294E-03	9.8575872E-03
80.50	-9.6318434E-01	-6.3034102E-03	-1.3495173E-02	-5.1668474E-03	1.1600090E-02
80.60	-9.9620360E-01	-7.5501480E-03	-1.1370205E-02	-3.9361130E-03	1.2946074E-02
80.70	-9.9521533E-01	-8.5643332E-03	-8.8571066E-03	-2.5925843E-03	1.3849589E-02
80.80	-9.6025327E-01	-9.3113577E-03	-6.0418935E-03	-1.1821842E-03	1.4279860E-02
80.90	-8.9251006E-01	-9.7657541E-03	-3.0208624E-03	2.4690326E-04	1.4222327E-02
81.00	-7.9429965E-01	-9.9120620E-03	1.0272338E-04	1.6458688E-03	1.3679131E-02
81.10	-6.6897414E-01	-9.7453467E-03	3.2221767E-03	2.9669501E-03	1.2669010E-02
81.20	-5.2081175E-01	-9.2713617E-03	6.2310359E-03	4.1650629E-03	1.1226636E-02
81.30	-3.5486860E-01	-8.5053421E-03	9.0266960E-03	5.1993381E-03	9.4014057E-03
81.40	-1.7681286E-01	-7.4764442E-03	1.1513896E-02	6.0345139E-03	7.2557323E-03
81.50	7.2787599E-03	-6.2168471E-03	1.3607946E-02	6.6421313E-03	4.8629017E-03

Table VIII
DIGITAL DATA - "STEADY STATE," q = 0.03

time	X_7	X_1	X_2	X_3	X_4
78.10	-4.3896002E-04	-6.1803157E-03	1.3382443E-02	6.5240934E-03	4.9182051E-03
78.20	1.8354077E-01	-4.7565051E-03	1.5014570E-02	6.8906488E-03	2.3925904E-03
78.30	3.6125495E-01	-3.1947102E-03	1.6134487E-02	6.9998631E-03	-2.1431597E-04
78.40	5.2663691E-01	-1.5482032E-03	1.6704126E-02	6.8480394E-03	-2.8136767E-03
78.50	6.7404101E-01	1.2686501E-04	1.6704133E-02	6.4403740E-03	-5.3169434E-03
78.60	7.9843644E-01	1.7733738E-03	1.6134523E-02	5.7907783E-03	-7.6388538E-03
78.70	8.9557413E-01	3.3351743E-03	1.5014668E-02	4.9214038E-03	-9.7003162E-03
78.80	9.6213949E-01	4.7589985E-03	1.3382651E-02	3.8618892E-03	-1.1431087E-02
78.90	9.9566017E-01	5.9962715E-03	1.1293985E-02	2.6483552E-03	-1.2772150E-02
79.00	9.9558487E-01	7.0047647E-03	8.8197470E-03	1.3221757E-03	-1.3677723E-02
79.10	9.6132299E-01	7.7500368E-03	6.0441777E-03	-7.1425449E-05	-1.4116818E-02
79.20	8.9424430E-01	8.2066113E-03	3.0618387E-03	-1.4849128E-03	-1.4074315E-02
79.30	7.9663869E-01	8.3588512E-03	-2.5590130E-05	-2.8700552E-03	-1.3551490E-02
79.40	6.7183811E-01	8.2014998E-03	-3.1127653E-03	-4.1795749E-03	-1.2565998E-02
79.50	5.2410129E-01	7.7398674E-03	-6.0942674E-03	-5.3687502E-03	-1.1151294E-02
79.60	3.5847457E-01	6.9896594E-03	-8.8682052E-03	-6.3969550E-03	-9.3555216E-03
79.70	1.8061056E-01	5.9764456E-03	-1.1339708E-02	-7.2290442E-03	-7.2398832E-03
79.80	-3.4189717E-03	4.7347952E-03	-1.3424191E-02	-7.8365607E-03	-4.8765693E-03
79.90	-1.8733180E-01	3.3375976E-03	-1.5050265E-02	-8.1987143E-03	-2.3462953E-03
80.00	-3.6465142E-01	1.7421156E-03	-1.6152212E-02	-8.3030981E-03	2.6445719E-04
80.10	-5.2991420E-01	9.3317198E-05	-1.6721912E-02	-8.1461180E-03	2.8664075E-03
80.20	-6.7658723E-01	-1.5829557E-03	-1.6710167E-02	-7.7331199E-03	5.3705451E-03
80.30	-8.0075327E-01	-3.2294160E-03	-1.6127368E-02	-7.0782082E-03	7.6911932E-03
80.40	-8.9723475E-01	-4.7898055E-03	-1.4993485E-02	-6.2037638E-03	9.7489593E-03
80.50	-9.6318434E-01	-6.2108039E-03	-1.3347370E-02	-5.1396770E-03	1.1473470E-02
80.60	-9.9620360E-01	-7.4438741E-03	-1.1245418E-02	-3.9223222E-03	1.2805788E-02
80.70	-9.9521533E-01	-8.4469140E-03	-8.7596099E-03	-2.5933090E-03	1.3700437E-02
80.80	-9.6025327E-01	-9.1856971E-03	-5.9750260E-03	-1.1980544E-03	1.4126949E-02
80.90	-8.9211006E-01	-9.6350389E-03	-2.9869120E-03	2.1577345E-04	1.4070897E-02
81.00	-7.9429965E-01	-9.7796508E-03	1.0259789E-04	1.5998880E-03	1.3534371E-02
81.10	-6.6897414E-01	-9.6146563E-03	3.1879855E-03	2.9070357E-03	1.2535879E-02
81.20	-5.2081175E-01	-9.1457489E-03	6.1639572E-03	4.0926096E-03	1.1109691E-02
81.30	-3.5486860E-01	-8.3839892E-03	8.9290340E-03	5.1161699E-03	9.3046455E-03
81.40	-1.7661286E-01	-7.3702508E-03	1.1389000E-02	5.9428202E-03	7.1824596E-03
81.50	7.2767599E-03	-6.1243291E-03	1.3460097E-02	6.5443925E-03	4.8156099E-03

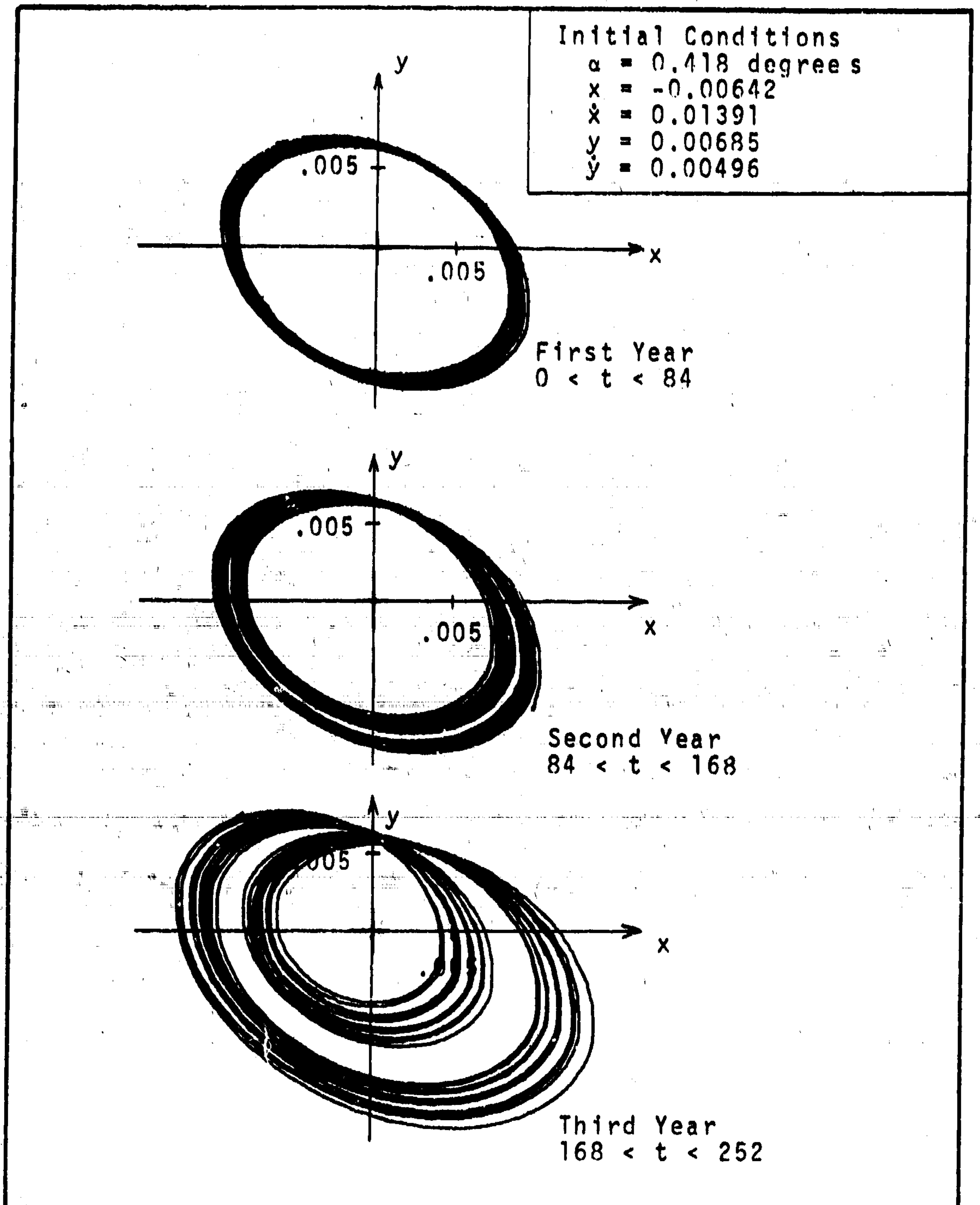


Figure 20

"NO-CONTROL" TRAJECTORY

V. Discussion

Model Deficiencies

The model used in this study is a gross simplification of the real world. The model does not account for such things as eccentricity of the moon's orbit, the pull of gravity from other planets, solar wind, etc. Even the nonlinearities of the very restricted four-body problem were removed by a linearization of the equations. However, since the control is generated from feedback of the state variables, it is felt that these model deficiencies are not serious.

Computational Approximations

Approximations were made on the linearized equations of motion which aided the computational process. All coefficients were rounded off to four significant figures and terms involving coefficients less than 0.00002 were neglected. The theory did not require that such approximations be made. State variables for $\cos \phi$ and $\sin \phi$ could have been defined which would handle the neglected terms. However, the previous work of Rudolph (Ref 2) proved that the approximations cause only minute error, undectable on an analog simulation.

Realizability of the Control System

Thrust Implimentation. The thrust levels required for a typical light-weight satellite are extremely small. As an

example, a 1000-lb. satellite requires an average thrust of about 0.0018 lbs. to confine its drift to a 50ⁿ mile radius (see Figure 5). Ion engines or monopropellant rockets would be likely candidat. for implimenting such low thrust levels. Higher thrust engines could also be used, if fired briefly at discrete time intervals. The thrust vector rotates with a period of about 15 days, therefore only two firings a day might approximate the continuous thrust vector with sufficient accuracy.

Rudolph proposed (Ref 2:49,74) that the necessary thrust be obtained from solar wind pressure. This scheme could not succeed, however, because the solar wind rotates (with respect to the ear. -moon coordinate system) at a frequency which is half that of the required thrust vector.

<u>Sensor Requirements</u>. The control system requires continuous knowledge of all seven state variables. This is not considered to be a difficult task because four of the states (x, y, $\dot{x}$, and $\dot{y}$) merely represent position and velocity. Two others ($\sin 2\phi$ and $\cos 2\phi$) can be determined by sensing the sun's direction. The only other state (X_5) is the constant, unity, however, the feedback gains associated with X_5, as well as all other feedback gains, must be calculated a priori, based on the best known values of astronomical constants.

<u>Attitude Control</u>. Although attitude control costs were not considered in this study, it is conceivable that a significant amount of fuel could be used to control attitude.

If the satellite mission does not have its own attitude requirements, it might be possible to approximate the proper rocket-nozzle orientation by establishing a spin of 4π radians per month, which is the rotation frequency of the thrust vector. If this scheme is used, control activation would be required only occasionally, if at all. The control would be needed merely to establish the initial spin and maintain synchronization.

VI. Summary and Recommendations

Summary

A position control system was devised for a space vehicle stationed at the earth-moon L4 libration point. The procedure was to assume the linearized equations of the very restricted four-body problem and then apply optimal control theory to a quadratic criterion. The optimal control was then modified, resulting in a control system which not only was less complex but also one that performed better with respect to the more meaningful criterion ΔV.

In studying the behavior of the controlled satellite, several interesting discoveries were made:

(1) The path about the L4 point becomes periodic after a brief transient.

(2) Initial conditions affect performance only during the transient.

(3) A stable orbit can be obtained without control by choosing proper initial conditions.

Recommendations

Further study is recommended to test the control system using a more realistic model of the real world. One possibility is to use the complete nonlinear equations of the very restricted four-body problem. Another possibility, and one which would be a far more complete test, is to use actual ephemeris data of the solar system.

Several other, less important, studies are recommended:

(1) Determine the effect of using impulse thrusting to approximate continuous thrust.

(2) Determine the effect of erroneous or noisy sensor information.

(3) Determine the effect of using inaccurate values of the astronomical constants.

(4) Determine the effect of solar wind.

Bibliography

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3. Siferd, R. E., "Some Periodic Trajectories in Cislunar Space - Very Restricted Four Body Approximation." GA/Mech/65-40, Thesis, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio: August 1965.

4. Ulshafer, P. M., "Libration Point Studies - An Analytical Approach." GGC/EE/63-14, Thesis, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio: June 1963.

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Appendix A

Fortran Program for Gain Calculations

Listed here is the digital program, written in Fortran IV, which solves for the gain matrix K. The program assumes a constant A matrix and outputs only the steady-state value of the K matrix. The gain matrix is computed for ten different values of q beginning with any specified value and incremented by any specified amount.

A time interval MG must be specified a priori, sufficiently large to insure that the steady-state value of K is reached. For five-place accuracy (five significant figures), MG must be at least 50 for a q of 0.01, 25 for a q of 1.0, and 12 for a q of 10. The value of the K matrix one time unit prior to the end time is outputed as a check to insure the steady-state value has been reached.

The 14 elements of the K matrix required for performance calculations are outputed on punch cards in the format required by the programs of Appendix B. The required input data is:

AA, AB, AC, AD, AE, AF, AG, AT, OM
the constants a, b, c, d, e, f, g, h, and w respectively

Q first q value

DQ q increment

MG time interval

The format of the input data must be as illustrated at the end of the program.

Gain Matrix Calculation

```
$JOB              0,19,750        67-040,STEINBERG,AFIT SE
$IBJOB
$IBFTC MAIN      M94,XR7
      COMMON /COM1/A(7,7),Q,GG(7,7),AT,OM,ALF
      DIMENSION Z(28),X(7)
      EXTERNAL FA
      NAMELIST /N1/AA,AB,AC,AD,AE,AF,AG,AT,OM
      NAMELIST /N4/Q,DQ,MG
      READ (5,N1)
      READ (5,N4)
      DT=-1.
      DO 6 I=1,7
      DO 6 J=1,7
 6    A(I,J)=0.
      A(1,2)=1.
      A(2,1)=AA
      A(2,3)=AB
      A(2,4)=2.
      A(2,5)=AD
      A(2,6)=AF
      A(2,7)=-AG
      A(3,4)=1.
      A(4,1)=AB
      A(4,2)=-2.
      A(4,3)=AC
      A(4,5)=AE
      A(4,6)=-AG
      A(4,7)=-AF
      A(6,7)=-2.*OM
      A(7,6)=2.*OM
      DO 52 NQ=1,10
      T=0.
      DO 12 I=1,28
 12   Z(I)=0.
      NT=0
 14   K=1
      DO 16 L=1,4
      DO 16 J=L,7
      GG(L,J)=Z(K)
      GG(J,L)=Z(K)
 16   K=K+1
      IF (NT.LT.MG-1) GO TO 71
      WRITE (6,13) Q,(GG(2,J),J=1,7),(GG(4,J),J=1,7)
 13   FORMAT (F10.4/(E17.8))
      IF (NT.EQ.MG) GO TO 72
 71   CALL RKDES (T,Z,28,DT,FA)
      NT=NT+1
      GO TO 14
 72   PUNCH 13,    Q,(GG(2,J),J=1,7),(GG(4,J),J=1,7)
```

```
      Q=Q+DQ
 52   CONTINUE
      STOP
      END
$IBFTC GAIN    M94,XR7
      SUBROUTINE FA (T,Z,DZ)
      COMMON /COM1/A(7,7),Q,GG(7,7),AT,OM,ALF
      DIMENSION Z(28),DZ(28),ZZ(7,7),ZA(7,7),ATZ(7,7),ZBZ(7,7)
      K=1
      DO 7 I=1,7
      DO 7 J=I,7
      ZZ(I,J)=Z(K)
      ZZ(J,I)=Z(K)
 7    K=K+1
      DO 3 I=1,7
      DO 3 J=I,7
      ZA(I,J)=0.
      ATZ(I,J)=0.
      ZBZ(I,J)=ZZ(I,2)*ZZ(2,J)+ZZ(I,4)*ZZ(4,J)
      DO 3 K=1,7
      ZA(I,J)=ZZ(I,K)*A(K,J)+ZA(I,J)
 3    ATZ(I,J)=A(K,I)*ZZ(K,J)+ATZ(I,J)
      K=1
      DO 11 I=1,7
      DO 11 J=I,7
      DZ(K)=-ZA(I,J)-ATZ(I,J)+ZBZ(I,J)
 11   K=K+1
      DZ(1)=DZ(1)-Q
      DZ(14)=DZ(14)-Q
      RETURN
      END
$IBFTC RUNKUT  M94,XR7
      SUBROUTINE RKDES(X,Y,N,DX,F)
      DIMENSION Y(28),YO(28),YT(28),YP(28),PO(28)
      DIMENSION P1(28),P2(28),P3(28)
 1    XO=X
      X=X+DX
      H=DX
 2    IF(ABS(H).GT.ABS(X-XO)) H=X-XO
      DO 4 I=1,N
 4    YO(I)=Y(I)
      HT=H
      XT=XO
      RMAXP = 1.E37
      DO 5 I=1,N
 5    YT(I)=YO(I)
      ASSIGN 6 TO K
      GO TO 20
 6    DO 7 I=1,N
 7    YP(I)=Y(I)
 8    HT=0.5*H
      ASSIGN 9 TO K
      GO TO 20
```

```
 9      DO 10 I=1,N
 10     YT(I)=Y(I)
        XT=XO+HT
        ASSIGN 11 TO K
 20     CALL F(XT,YT,PO)
        DO 21 I=1,N
 21     Y(I)=YT(I)+0.5*HT*PO(I)
        CALL F(XT+0.5*HT,Y,P1)
        DO 22 I=1,N
 22     Y(I)=YT(I)+.5*HT*P1(I)
        CALL F(XT+0.5*HT,Y,P2)
        DO 23 I=1,N
 23     Y(I)=YT(I)+HT*P2(I)
        CALL F(XT+HT,Y,P3)
        DO 24 I=1,N
 24     Y(I)=YT(I)+HT*(PO(I)+2.*(P1(I)+P2(I))+P3(I))/6.
        GO TO K,(6,9,11)
 11     RMAX=0.
        DO 12 I=1,N
 12     RMAX=AMAX1(RMAX,.07*ABS((Y(I)-YP(I))/Y(I)))
        IF ((RMAX.GT.1.E-06).AND.(RMAX.LT.RMAXP)) GO TO 17
        XO=XO+H
        IF(XO.EQ.X) RETURN
        IF((RMAX.LT.1.E-7).OR.(RMAX.GT.RMAXP)) H=H+H
        GO TO 2
 17     H=HT
        XT=XO
        DO 19 I=1,N
 18     YP(I)=YT(I)
 19     YT(I)=YO(I)
        RMAXP = RMAX
        GO TO 8
        END
$DATA
 $N1 AA=.7528,AB=1.267,AC=2.253,AD=.001365,
  AE=.002423,  AF=.004094,  AG=.007268,  AT=.008393,
  OM=.9252  $
 $N4  Q=1.,  DQ=1.,  MG=25  $
$EOF
```

Appendix B

Fortran Program for Performance Calculations

Two programs are listed here which calculate performance. They differ only in the type of data which is outputed. The first program outputs cost data, i.e., ΔV, peak drift, control and drift. The second program outputs the states X_7, X_1, X_2, X_3 and X_4. Both programs calculate performance for a one-year mission. The two programs are basically the same and could easily be combined into a single program.

The required input data for either program is:

AA, AB, AC, AD, AE, AF, AG, AT, OM
the constants a, b, c, d, e, f, g, h, and w respectively

X(1), X(2), X(3), X(4)
the initial values of states X_1, X_2, X_3 and X_4 respectively

ALF the initial sun direction

LM the number of q's for which the performance is to be calculated (LM is restricted to ten or less)

LM sets of gains as outputed from the program of Appendix A

The format of the input data must be as illustrated at the ends of the two programs.

Performance Calculation - Costs

```
$JOB            0,15,5000       67-114,STEINBERG,AFIT SE
$IBJOB
$IBFTC MAIN     M94,XR7
       COMMON /COM1/A(7,7),  GG(7,7),AT,OM,ALF
       DIMENSION Z(28),X(7),Q(10),G3(4,7,10),XX(4)
       DIMENSION V2(841),RP2(841),R2(841),T2(841),UM(841)
       DOUBLE PRECISION DV
       EXTERNAL FX
       NAMELIST /N1/AA,AB,AC,AD,AE,AF,AG,AT,OM
       NAMELIST /N3/X,ALF,LM
       READ (5,N1)
       READ (5,N3)
       DO 99 J=1,4
 99    XX(J)=X(J)
       LP=LM+1
       Q(LP)=0.
       DO 43 K=2,4,2
       DO 43 J=1,7
 43    G3(K,J,LP)=0.
       DO 42 NQ=1,LM
       READ (5,13) Q(NQ),((G3(K,J,NQ),J-1,7),K-2,4,2)
 42    WRITE(6,13) Q(NQ),((G3(K,J,NQ),J=1,7),K=2,4,2)
       WRITE(6,13) Q(LP),((G3(K,J,LP),J=1,7),K=2,4,2)
 13    FORMAT (F10.4/(E17.8))
       ALFD=ALF
       ALF=ALF*3.1415927/180.
       DO 6 I=1,7
       DO 6 J=1,7
 6     A(I,J)=0.
       A(1,2)=1.
       A(2,1)=AA
       A(2,3)=AB
       A(2,4)=2.
       A(2,5)=AD
       A(2,6)=AF
       A(2,7)=-AG
       A(3,4)=1.
       A(4,1)=AB
       A(4,2)=-2.
       A(4,3)=AC
       A(4,5)=AE
       A(4,6)=-AG
       A(4,7)=-AF
       A(6,7)=-2.*OM
       A(7,6)=2.*OM
       TF=84
       NF=10.*TF+1.1
       DT=.01
       DO 82 NQ=1,LP
```

```
      DO 83 LA=2,4,2
      DO 83 LB=1,7
 83   GG(LA,LB)=G3(LA,LB,NQ)
      DO 62 J=1,4
 62   X(J)=XX(J)
      DV=0.
      T=0.
      RPO=0.
      DM1=2.*(OM*T+ALF)
      X(5)=1.
      X(6)=COS(DM1)
      X(7)=SIN(DM1)
      UX=0.
      UY=0.
      DO 24 L=1,7
      UX=UX-GG(2,L)*X(L)
 24   UY=UY-GG(4,L)*X(L)
      USQ=UX**2+UY**2
      U=SQRT(USQ)
      DV=DV+U*DT
      RSQ=X(1)**2+X(3)**2
      R=SQRT(RSQ)
      RP=AMAX1(R,RPO)
      RPO=RP
      NA=1
      V2(NA)=DV
      RP2(NA)=RPO
      R2(NA)=R
      UM(NA)=U
      T2(NA)=T
      WRITE (6,14) NQ,Q(NQ)
 14   FORMAT (1H1,10X,3HJ =,I3,10X,3HQ =,F8.4///17X,5HPOINT,
     24X,4HTIME,9X,7HDELTA V,6X,10HPEAK DRIFT,11X,5HDRIFT,6X,
     37HCONTROL/2X)
      DO 54 NA=2,NF
      DO 53 NB=1,10
      CALL RKDES (T,X,4,DT,FX)
 31   DM1=2.*(OM*T+ALF)
      X(5)=1.
      X(6)=COS(DM1)
      X(7)=SIN(DM1)
      UX=0.
      UY=0.
      DO 22 L=1,7
      UX=UX-GG(2,L)*X(L)
 22   UY=UY-GG(4,L)*X(L)
      USQ=UX**2+UY**2
      U=SQRT(USQ)
      DV=DV+U*DT
      RSQ=X(1)**2+X(3)**2
      R=SQRT(RSQ)
      RP=AMAX1(R,RPO)
 53   RPO=RP
```

```
      V2(NA)=DV
      RP2(NA)=RPO
      R2(NA)=R
      UM(NA)=U
 54   T2(NA)=T
 82   WRITE (6,15) (L,T2(L),V2(L),RP2(L),R2(L),UM(L),L=1,NF)
 15   FORMAT (I22,0PF8.2,1PE16.7,1PE16.7,1PE16.7,1PE16.7)
      STOP
      END
$IBFTC STATE    M94,XR7
      SUBROUTINE FX (T,X,DX)
      COMMON /COM1/A(7,7),  GG(7,7),AT,OM,ALF
      DIMENSION X(7),DX(4),AX(4,7)
      DM1=2.*(OM*T+ALF)
      X(5)=1.
      X(6)=COS(DM1)
      X(7)=SIN(DM1)
      A21=A(2,1)
      A23=A(2,3)
      A41=A(4,1)
      A43=A(4,3)
      DM2=AT*X(6)
      DM3=AT*X(7)
      A(2,1)=A(2,1)+DM2
      A(2,3)=A(2,3)-DM3
      A(4,1)=A(4,1)-DM3
      A(4,3)=A(4,3)-DM2
  8   DO 3 L=1,7
      AX(1,L)=0.
  3   AX(3,L)=0.
      AX(1,2)=1.
      AX(3,4)=1.
      DO 4 L=1,7
      AX(2,L)=A(2,L)-GG(2,L)
  4   AX(4,L)=A(4,L)-GG(4,L)
      A(2,1)=A21
      A(2,3)=A23
      A(4,1)=A41
      A(4,3)=A43
      DO 5 L=1,4
      DX(L)=0.
      DO 5 K=1,7
  5   DX(L)=AX(L,K)*X(K)+DX(L)
      RETURN
      END
$IBFTC RUNKUT   M94,XR7
      SUBROUTINE RKDES(X,Y,N,DX,F)
      DIMENSION Y(28),Y0(28),YT(28),YP(28),P0(28)
      DIMENSION P1(28),P2(28),P3(28)
  1   X0=X
      X=X+DX
      H=DX
  2   IF(ABS(H).GT.ABS(X-X0)) H=X-X0
```

```
      DO 4 I=1,N
4     YO(I)=Y(I)
      HT=H
      XT=XO
      RMAXP = 1.E37
      DO 5 I=1,N
5     YT(I)=YO(I)
      ASSIGN 6 TO K
      GO TO 20
6     DO 7 I=1,N
7     YP(I)=Y(I)
8     HT=0.5*H
      ASSIGN 9 TO K
      GO TO 20
9     DO 10 I=1,N
10    YT(I)=Y(I)
      XT=XO+HT
      ASSIGN 11 TO K
20    CALL F(XT,YT,PO)
      DO 21 I=1,N
21    Y(I)=YT(I)+0.5*HT*PO(I)
      CALL F(XT+0.5*HT,Y,P1)
      DO 22 I=1,N
22    Y(I)=YT(I)+.5*HT*P1(I)
      CALL F(XT+0.5*HT,Y,P2)
      DO 23 I=1,N
23    Y(I)=YT(I)+HT*P2(I)
      CALL F(XT+HT,Y,P3)
      DO 24 I=1,N
24    Y(I)=YT(I)+HT*(PO(I)+2.*(P1(I)+P2(I))+P3(I))/6.
      GO TO K,(6,9,11)
11    RMAX=0.
      DO 12 I=1,N
12    RMAX=AMAX1(RMAX,.07*ABS((Y(I)-YP(I))/Y(I)))
      IF ((RMAX.GT.1.E-06).AND.(RMAX.LT.RMAXP)) GO TO 17
      XO=XO+H
      IF(XO.EQ.X) RETURN
      IF((RMAX.LT.1.E-7).OR.(RMAX.GT.RMAXP)) H=H+H
      GO TO 2
17    H=HT
      XT=XO
      DO 19 I=1,N
18    YP(I)=YT(I)
19    YT(I)=YO(I)
      RMAXP = RMAX
      GO TO 8
      END
 $DATA
  $N1 AA=.7528,AB=1.267,AC=2.253,AD=.001365,
   AE=.002423,  AF=.004094,  AG=.007268,  AT=.008393,
   OM=.9252  $
  $N3 X(1)=.002, X(2)=0.,  X(3)=0., X(4)=0., ALF=90.,
   LM=4  $
```

```
  0.0300
-0.72882692E 00
 0.63458543E 00
-0.12758727E 01
-0.53724663E 00
-0.[illegible]3760410E-02
-0.50880010E-03
-0.51252145E-02
 0.11846696E 01
-0.53724663E 00
 0.17579287E 01
 0.10383819E 01
 0.19819423E-02
-0.42777915E-02
 0.59098098E-02
  0.3000
-0.48292343E-00
 0.82325358E 00
-0.12893765E 01
-0.46494085E-00
-0.11963367E-02
-0.21485960E-03
-0.75401255E-02
 0.22391029E 01
-0.46494085E-00
 0.[illegible]9210094E 01
 0.19406404E 01
 0.33821663E-02
-0.11317102E-01
 0.42906228E-02
  3.0000
 0.95098654E 00
 0.14873070E 01
-0.11995051E 01
-0.79091149E-01
-0.62572880E-04
 0.27161561E-02
-0.96700978E-02
 0.35325503E 01
-0.79091149E-01
 0.41497470E 01
 0.28243360E 01
 0.43564986E-02
-0.12124909E-01
-0.20369150E-02
 10.0000
 0.25984848E 01
 0.21933012E 01
-0.14953858E 01
 0.94373211E-01
 0.71278690E-03
 0.40733601E-02
-0.88293092E-02
```

```
  0.42276491E 01
  0.94373211E-01
  0.52614085E 01
  0.33002126E 01
  0.42135844E-02
 -0.10761255E-01
 -0.32459094E-02
$EOF
```

Performance Calculation, States

```
$JOB            0,15,5000        67-114,STEINBERG,AFIT SE
$IBJOB
$IBFTC MAIN     M94,XR7
      COMMON /COM1/A(7,7),   GG(7,7),AT,OM,ALF
      DIMENSION Z(28),X(7),Q(10),G3(4,7,10),XX(4)
      DIMENSION V2(841),RP2(841),R2(841),T2(841),UM(841)
      DIMENSION X7(841),X1(841),X2(841),X3(841),X4(841)
      DOUBLE PRECISION DV
      EXTERNAL FX
      NAMELIST /N1/AA,AB,AC,AD,AE,AF,AG,AT,OM
      NAMELIST /N3/X,ALF,LM
      READ (5,N1)
      READ (5,N3)
      DO 99 J=1,4
 99   XX(J)=X(J)
      LP=LM+1
      Q(LP)=0.
      DO 43 K=2,4,2
      DO 43 J=1,7
 43   G3(K,J,LP)=0.
      DO 42 NQ=1,LM
      READ (5,13) Q(NQ),((G3(K,J,NQ),J=1,7),K=2,4,2)
 42   WRITE(6,13) Q(NQ),((G3(K,J,NQ),J=1,7),K=2,4,2)
      WRITE(6,13) Q(LP),((G3(K,J,LP),J=1,7),K=2,4,2)
 13   FORMAT (F10.4/(E17.8))
      ALFD=ALF
      ALF=ALF*3.1415927/180.
      DO 6 I=1,7
      DO 6 J=1,7
 6    A(I,J)=0.
      A(1,2)=1.
      A(2,1)=AA
      A(2,3)=AB
      A(2,4)=2.
      A(2,5)=AD
      A(2,6)=AF
      A(2,7)=-AG
      A(3,4)=1.
      A(4,1)=AB
      A(4,2)=-2.
      A(4,3)=AC
      A(4,5)=AE
      A(4,6)=-AG
      A(4,7)=-AF
      A(6,7)=-2.*OM
      A(7,6)=2.*OM
      TF=84
      NF=10.*TF+1.1
      DT=.01
```

```
      DO 82 NQ=1,LP
      DO 83 LA=2,4,2
      DO 83 LB=1,7
83    GG(LA,LB)=G3(LA,LB,NQ)
      DO 62 J=1,4
62    X(J)=XX(J)
      DV=0.
      T=0.
      RPO=0.
      DM1=2.*(OM*T+ALF)
      X(5)=1.
      X(6)=COS(DM1)
      X(7)=SIN(DM1)
      UX=0.
      UY=0.
      DO 24 L=1,7
      UX=UX-GG(2,L)*X(L)
24    UY=UY-GG(4,L)*X(L)
      USQ=UX**2+UY**2
      U=SQRT(USQ)
      DV=DV+U*DT
      RSQ=X(1)**2+X(3)**2
      R=SQRT(RSQ)
      RP=AMAX1(R,RPO)
      RPO=RP
      NA=1
      V2(NA)=DV
      RP2(NA)=RPO
      R2(NA)=R
      UM(NA)=U
      X7(NA)=X(7)
      X1(NA)=X(1)
      X2(NA)=X(2)
      X3(NA)=X(3)
      X4(NA)=X(4)
      T2(NA)=T
      WRITE (6,14) NQ,Q(NQ)
14    FORMAT (1H1,10X,3HJ =,I3,10X,3HQ =,F8.4///17X,5HPOINT,
     24X,4HTIME,9X,2HX7,14X,2HX1,14X,2HX2,14X,2HX3,14X,2HX4/2X)
      DO 54 NA=2,NF
      DO 53 NB=1,10
      CALL RKDES (T,X,4,DT.FX)
31    DM1=2.*(OM*T+ALF)
      X(5)=1.
      X(6)=COS(DM1)
      X(7)=SIN(DM1)
      UX=0.
      UY=0.
      DO 22 L=1,7
      UX=UX-GG(2,L)*X(L)
22    UY=UY-GG(4,L)*X(L)
      USQ=UX**2+UY**2
      U=SQRT(USQ)
```

```
      DV=DV+U*DT
      RSQ=X(1)**2+X(3)**2
      R=SQRT(RSQ)
      RP=AMAX1(R,RPO)
   53 RPO=RP
      V2(NA)=DV
      RP2(NA)=RPO
      R2(NA)=R
      UM(NA)=U
      X7(NA)=X(7)
      X1(NA)=X(1)
      X2(NA)=X(2)
      X3(NA)=X(3)
      X4(NA)=X(4)
   54 T2(NA)=T
   82 WRITE (6,15) (L,T2(L),X7(L),X1(L),X2(L),X3(L),X4(L),
     2L=1,NF)
   15 FORMAT (I22,0PF8.2,1P5E16.7)
      STOP
      END
$IBFTC STATE    M94,XR7
      SUBROUTINE FX (T,X,DX)
      COMMON /COM1/A(7,7),   GG(7,7),AT,OM,ALF
      DIMENSION X(7),DX(4),AX(4,7)
      DM1=2.*(OM*T+ALF)
      X(5)=1.
      X(6)=COS(DM1)
      X(7)=SIN(DM1)
      A21=A(2,1)
      A23=A(2,3)
      A41=A(4,1)
      A43=A(4,3)
      DM2=AT*X(6)
      DM3=AT*X(7)
      A(2,1)=A(2,1)+DM2
      A(2,3)=A(2,3)-DM3
      A(4,1)=A(4,1)-DM3
      A(4,3)=A(4,3)-DM2
    8 DO 3 L=1,7
      AX(1,L)=0.
    3 AX(3,L)=0.
      AX(1,2)=1.
      AX(3,4)=1.
      DO 4 L=1,7
      AX(2,L)=A(2,L)-GG(2,L)
    4 AX(4,L)=A(4,L)-GG(4,L)
      A(2,1)=A21
      A(2,3)=A23
      A(4,1)=A41
      A(4,3)=A43
      DO 5 L=1,4
      DX(L)=0.
      DO 5 K=1,7
```

```
 5     DX(L)=AX(L,K)*X(K)+DX(L)
       RETURN
       END
$IBFTC RUNKUT  M94,XR7
       SUBROUTINE RKDES(X,Y,N,DX,F)
       DIMENSION Y(28),YO(28),YT(28),YP(28),PO(28)
       DIMENSION P1(28),P2(28),P3(28)
 1     XO=X
       X=X+DX
       H=DX
 2     IF(ABS(H).GT.ABS(X-XO)) H=X-XO
       DO 4 I=1,N
 4     YO(I)=Y(I)
       HT=H
       XT=XO
       RMAXP = 1.E37
       DO 5 I=1,N
 5     YT(I)=YO(I)
       ASSIGN 6 TO K
       GO TO 20
 6     DO 7 I=1,N
 7     YP(I)=Y(I)
 8     HT=0.5*H
       ASSIGN 9 TO K
       GO TO 20
 9     DO 10 I=1,N
 10    YT(I)=Y(I)
       XT=XO+HT
       ASSIGN 11 TO K
 20    CALL F(XT,YT,PO)
       DO 21 I=1,N
 21    Y(I)=YT(I)+0.5*HT*PO(I)
       CALL F(XT+0.5*HT,Y,P1)
       DO 22 I=1,N
 22    Y(I)=YT(I)+.5*HT*P1(I)
       CALL F(XT+0.5*HT,Y,P2)
       DO 23 I=1,N
 23    Y(I)=YT(I)+HT*P2(I)
       CALL F(XT+HT,Y,P3)
       DO 24 I=1,N
 24    Y(I)=YT(I)+HT*(PO(I)+2.*(P1(I)+P2(I))+P3(I))/6.
       GO TO K,(6,9,11)
 11    RMAX=0.
       DO 12 I=1,N
 12    RMAX=AMAX1(RMAX,.07*ABS((Y(I)-YP(I))/Y(I)))
       IF ((RMAX.GT.1.E-06).AND.(RMAX.LT.RMAXP)) GO TO 17
       XO=XO+H
       IF(XO.EQ.X) RETURN
       IF((RMAX.LT.1.E-7).OR.(RMAX.GT.RMAXP)) H=H+H
       GO TO 2
 17    H=HT
       XT=XO
       DO 19 I=1,N
```

```
 18    YP(I)=YT(I)
 19    YT(I)=YO(I)
       RMAXP = RMAX
       GO TO 8
       END
$DATA
 $N1 AA=.7528,AB=1.267,AC=2.253,AD=.001365,
  AE=.002423,   AF=.004094,   AG=.007268,   AT=.008393,
  OM=.9252   $
 $N3 X(1)=.004,   X(2)=0.,   X(3)=0.,  X(4)=0.,  ALF=45.,
  LM=3   $
    0.0100
  -0.65117057E 00
   0.55882413E 00
  -0.11179198E 01
  -0.45553420E-00
  -0.12170684E-02
  -0.74567784E-03
  -0.42003048E-02
   0.80780128E 00
  -0.45553420E-00
   0.12512737E 01
   0.68767665E 00
   0.13943277E-02
  -0.19536840E-02
   0.46583379E-02
    0.0200
  -0.70958859E 00
   0.60904848E 00
  -0.12282993E 01
  -0.51343259E 00
  -0.13313749E-02
  -0.58913541E-03
  -0.47957293E-02
   0.10358803E 01
  -0.51343259E 00
   0.15639579E 01
   0.90031803E 00
   0.17550997E-02
  -0.32934009E-02
   0.55174362E-02
    0.0300
  -0.72882692E 00
   0.63458543E 00
  -0.12758727E 01
  -0.53724663E 00
  -0.13760410E-02
  -0.50880010E-03
  -0.51252145E-02
   0.11846696E 01
  -0.53724663E 00
   0.17579287E 01
   0.10383819E 01
```

```
   0.19819423E-02
  -0.42777915E-02
   0.59098098E-02
$EOF
```

Appendix C

Analog Diagrams

The analog simulation was performed on two EAI TR 48's coupled together. One computer alone did not have a sufficient number of amplifiers or potentiometers to simulate the entire system. The circuits are illustrated in Figures 21 thru 24. For the correct units, all indicated variables are measured in volts, e.g. 4 volts for the variable 100x implies that x is equal to 0.04. Time scaling was unnecessary.

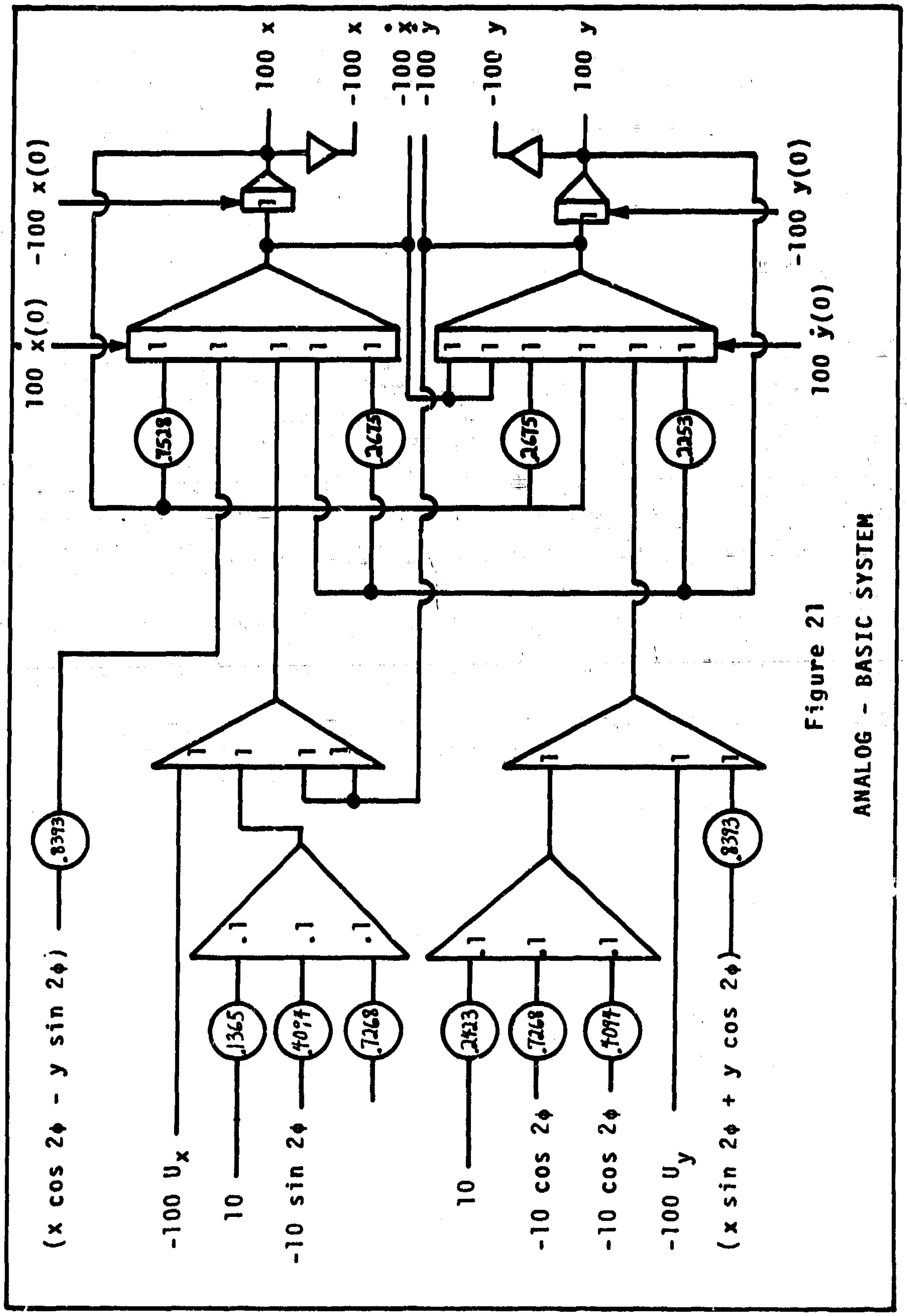
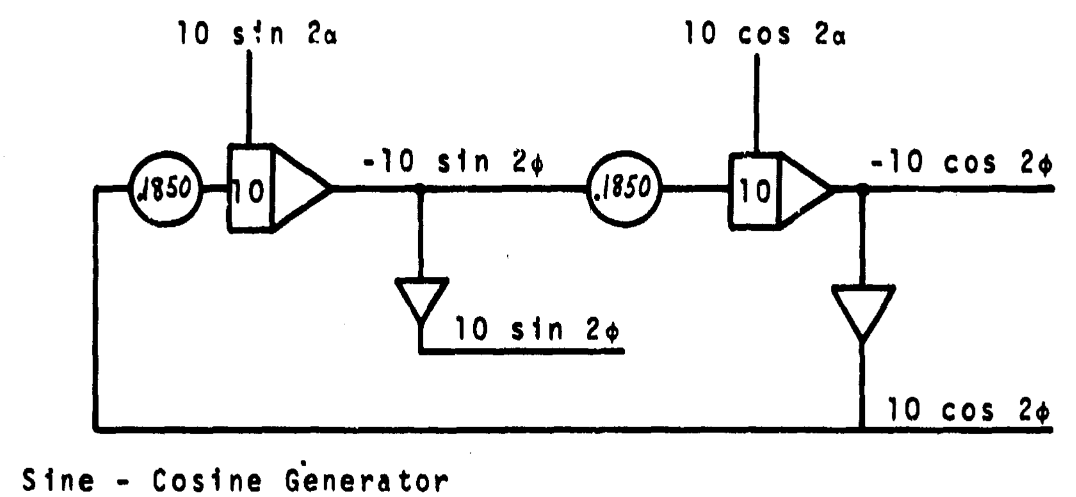


Figure 21

ANALOG - BASIC SYSTEM



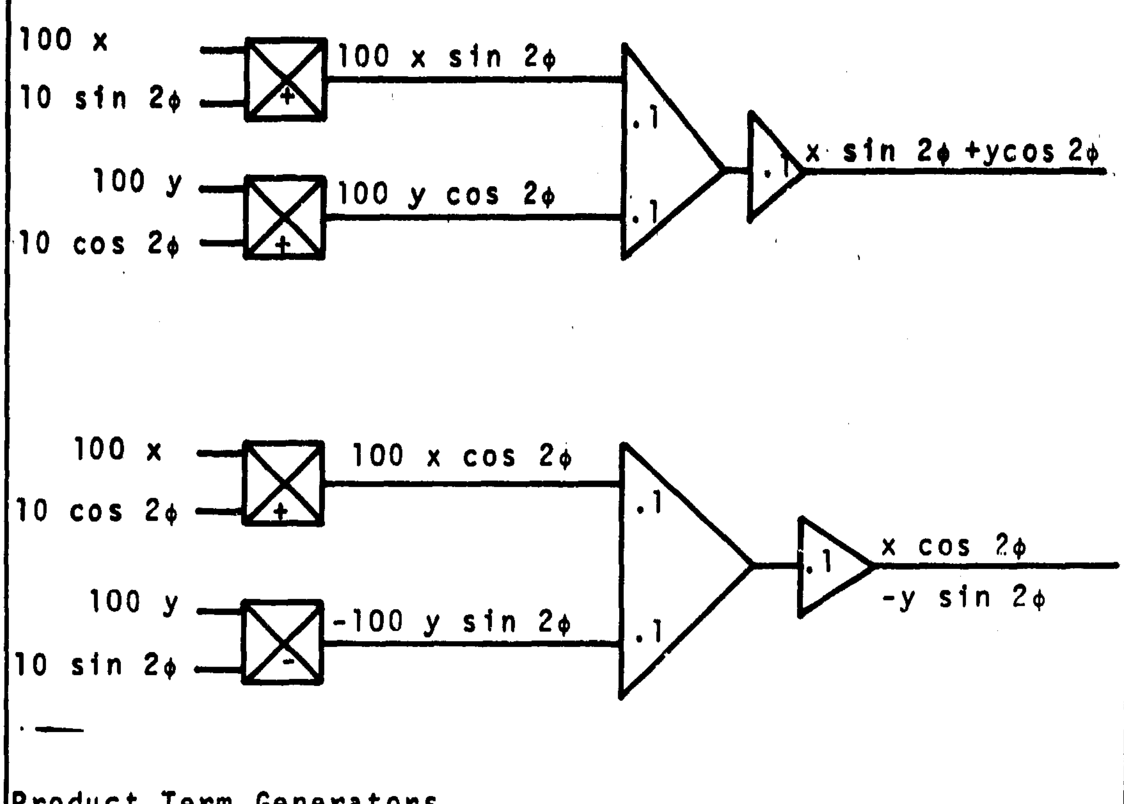


Figure 22

AUXILIARY CIRCUITS

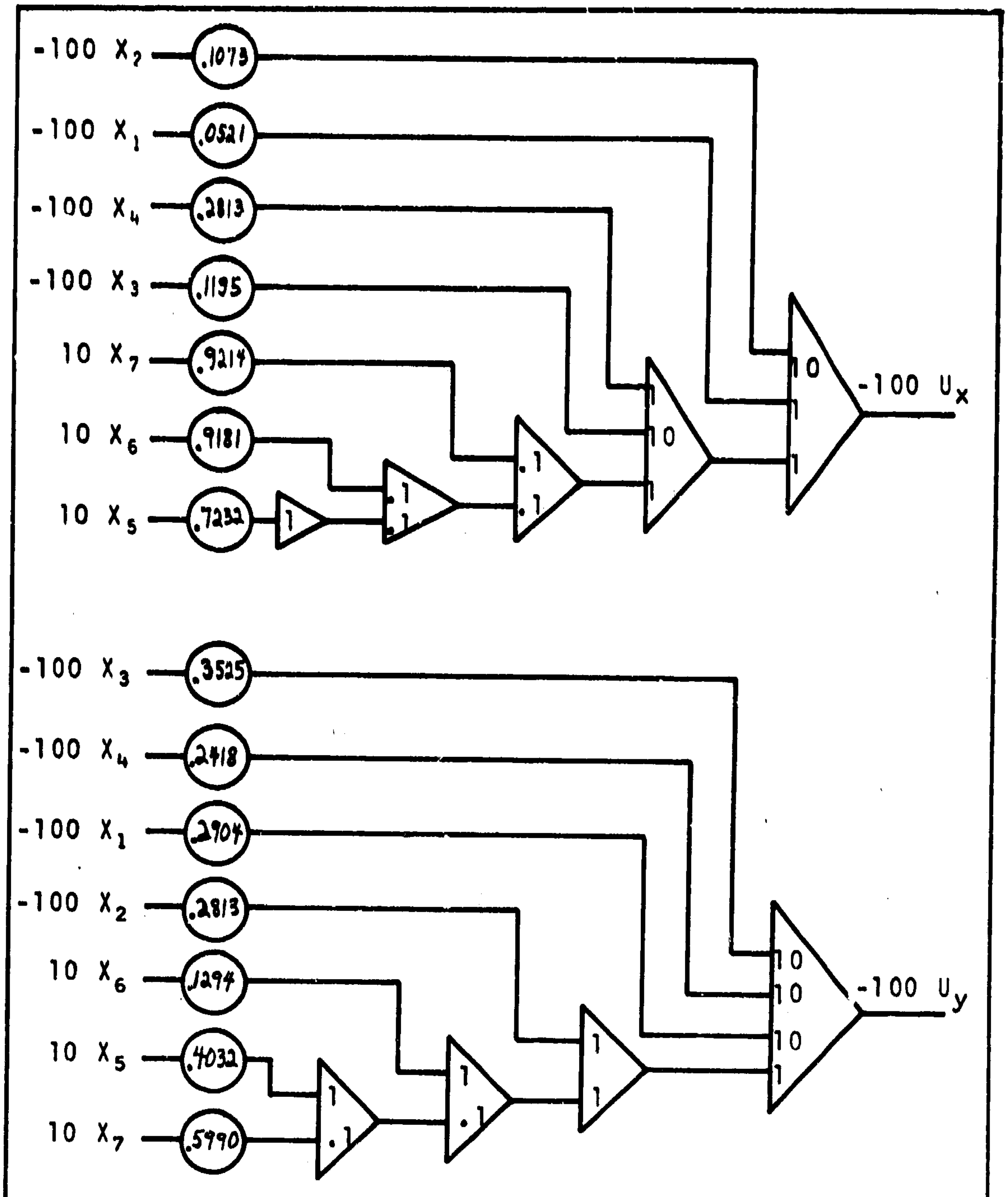


Figure 23

ANALOG - CONTROL GENERATOR, q=1

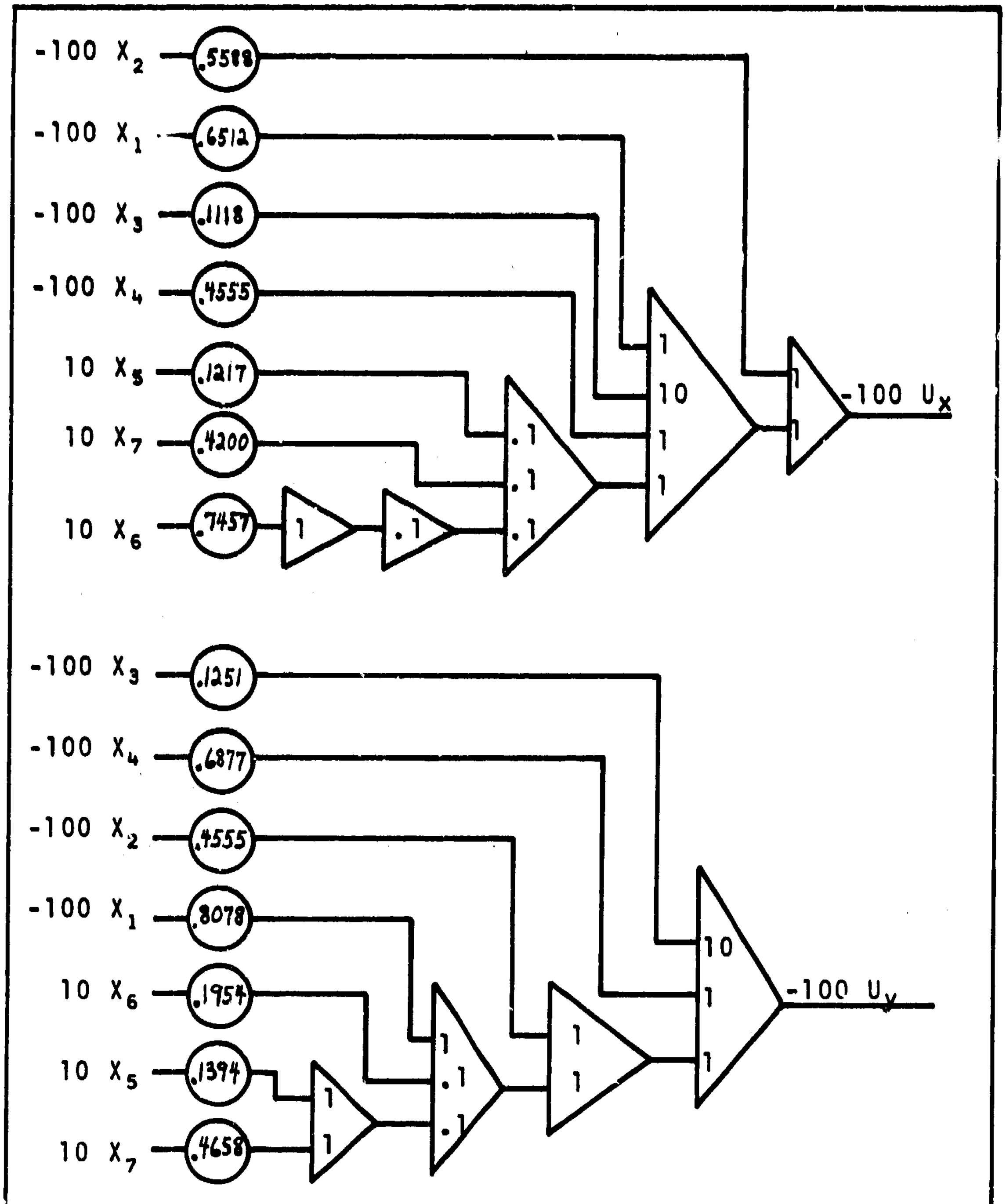


Figure 24

ANALOG - CONTROL GENERATOR, q=0.01

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13. ABSTRACT

A control system is devised for maintaining a space vehicle in close proximity to one of the earth-moon triangular libration points while minimizing fuel consumption. The problem is formulated as an optimal state regulator problem of variational calculus and then modern control theory is applied. A quadratic performance criterion is used which leads to a linear feedback control system. The feedback gains are obtained by solving a matrix differential equation of the Ricatti type on a high-speed digital computer. Performance of the resulting optimal system is verified and further analyzed on an analog computer. (U)

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14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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